Combinatorial Discrete Choice: Theory and Application to Multinational Production*

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Discrete choice problems with complementarities among options quickly grow infeasible to solve, since they generically require evaluating all *combinations* of choices. We develop a solution method that applies whenever choices are weakly complementary or substitutable, using the implied choice monotonicity to discard suboptimal combinations without computing their payoff. It is orders of magnitude faster than existing approaches, finds the global solution, and extends to heterogeneous-agent settings. Using our method, we calibrate a general equilibrium model of multinational firms selecting global production locations to show that complementarities among locations can amplify, dampen, or even reverse the welfare gains from multinational production.

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1. Introduction

Complementarities among discrete choices typically force decision-makers to evaluate all possible *combinations*. The resulting dimensionality renders such combinatorial discrete choice problems practically infeasible to solve without additional structure—even with a moderate number of options. Yet, complementarities arise naturally from network, scale, or cannibalization effects in economic settings, such as multinational firms choosing where to operate or retailers selecting store locations.

We develop a unified method to solve combinatorial discrete choice problems (CDCPs) whose objective functions satisfy a simple single-crossing condition, enforcing a weak form of complementarity or substitutability. Our approach extends to heterogeneous-agent settings where payoffs depend on type, is many orders of magnitude faster than existing methods, and always finds the global optimum. We apply it to a quantitative general equilibrium model of multinational production, in which heterogeneous firms choose global production locations. The firm problem is a CDCP: complementarities arise from scale and cannibalization effects, and fixed costs make location choices discrete. Using the calibrated model, we show that incorporating complementarities and fixed costs—enabled by our method—materially affects the estimated welfare gains from multinational production.

Our method builds on Jia (2008), which develops the insight that positive complementarities induce a form of *choice monotonicity* that can be used to iteratively discard suboptimal choice combinations without computing their payoff. If an option yields a negative payoff even when all other options are selected—maximizing complementarities—it cannot be part of the optimal combination. Likewise, if an option has a positive payoff even in isolation, it must be included.

We show that the elimination logic in Jia (2008) extends beyond the case of positive complementarities to problems satisfying a weaker single-crossing condition. This condition requires only that if a choice yields a positive payoff as part of a given combination, its payoff remains weakly positive either when additional choices are added or when existing ones are removed. Notably, our single crossing condition nests the case of negative complementarities—for example, when new locations cannibalize demand—for which few solution methods exist, and to which the elimination logic was previously assumed not to apply.

Iteratively applying the elimination logic often dramatically reduces the set of potentially optimal combinations, but may still leave multiple candidates. In such cases, Jia (2008) applies the brute force approach of explicitly computing the payoff of all remaining combinations. Instead, we introduce a recursive branching procedure that reapplies the elimination logic to subsets of the remaining candidates until the optimal combination is found. Our branching method collapses to brute force in the worst case only.

CDCPs with heterogeneous agents pose an added challenge: payoffs vary by agent type, so the solution for one type need not apply to others. The incumbent approach—discretizing the type space and solving a CDCP at each grid point—may introduce interpolation error. We further extend our method to recover the exact *policy function* mapping types to optimal combinations of choices. The extension requires a second single-crossing condition that induces a form of *type monotonicity*, allowing us to identify the cutoff types where the policy function changes value without solving the CDCP for every type.

We apply our method to solve a general equilibrium model of multinational production where heterogeneous firms select locations for production. The firm problem is a CDCP: scale economies and demand cannibalization across production locations give rise to complementarities, while location-specific fixed costs make location choices discrete. In addition, firm heterogeneity implies that optimal location combinations differ across firms. We show how our single crossing conditions correspond to explicit parameter restrictions in the model.

We calibrate the model to match trade and multinational activity across 32 countries. Because production location sets vary across firms, the model does not yield the aggregate log-linear estimating equations used in standard gravity-based calibrations. Instead, we solve the full model repeatedly to match aggregate flows, computing the policy function that maps firm productivity to optimal location choices at each iteration. Our method makes this strategy feasible even with many countries, arbitrary firm heterogeneity, and a wide range of complementarities—from strongly positive to strongly negative.

We benchmark the performance of our method using up to 256 synthetic countries generated by sampling from the empirical distributions of productivity and cost parameters obtained from our calibration. With positive complementarities, our method is up to four orders of magnitude faster than brute-force and an order of magnitude faster than the method in Jia (2008) with discretized firm heterogeneity. It performs equally fast with negative complementarities for which brute force was the only incumbent solution method. The speed of our method matters especially in general equilibrium settings, where policy functions must be solved repeatedly.

To avoid the computational burden of solving CDCPs, many quantitative models of multinational production rule them out by abstracting from mechanisms that give rise either to complementarities, such as scale economies or cannibalization, or to discrete choices, such as fixed costs. We assess the role of these ingredients in shaping the counterfactual welfare predictions of the full model. To guide interpretation, we derive a new expression for the gains from multinational production that extends the welfare formula in Arkolakis, Costinot, and Rodríguez-Clare (2012) to our setting with scale economies, cannibalization, and fixed costs. Our formula decomposes the gains from multinational production into intuitive channels, including one that captures the effective productivity gains unlocked by firms that operate multiple global production locations.

Quantitatively, both complementarities and fixed costs shape the gains from multinational production. Countries with many multinationals benefit significantly, as international production reduces marginal costs and generates large profits. In contrast, less productive countries with few multinationals may lose: few of their firms generate profit abroad, while they play host to the local affiliates of foreign firms that crowd out domestic producers. Positive complementarities amplify this dynamic by creating increasing returns for multinationals, further boosting home-country gains and intensifying the crowding out in host countries. On the other hand, negative complementarities temper this effect. Calibrations that omit fixed costs overstate welfare gains, because all firms participate in the productivity gains accessible through multinational production.

Modeling complementarities and agent heterogeneity has long posed a challenge in the discrete choice literature. The classic random utility framework of McFadden (1974) assumes perfect substitutability, so that each agent chooses a single option. This assumption permits rich multidimensional heterogeneity via option-specific shocks and yields closed-form aggregation. However, the literature has struggled to allow for imperfect substitutability or complementarity among options. Some authors allow agents to choose multiple options at once but without interdependencies in payoffs (see, e.g., Hendel 1999). Others associate each combination of options with a random utility component (see, e.g., Train, McFadden, and Ben-Akiva 1987; Gentzkow 2007), but aggregation then requires evaluating payoffs for all combinations which quickly becomes infeasible in practice.

Our paper contributes to a growing quantitative literature on combinatorial discrete choice problems that abstract from random utility components, allowing instead for large numbers of options, more flexible functional forms, and complementarities. Jia (2008) introduces a foundational method for solving CDCPs with strongly positive complementarities, often applied to store expansion and firm sourcing problems (see Antras, Fort, and Tintelnot 2017; Alfaro-Urena et al. 2023; Antràs et al. 2024b). Our extension to negative complementarities has similarly been applied in recent work (see, e.g., Jiang 2023; Liu 2023; Sabal 2025). By developing a method that handles both positive and negative complementarities, we also lay the groundwork to solve mixed-complementarity problems that arise, for example, in the multinational production model of Antràs et al. (2024b). Castro-Vincenzi et al. (2024) build on our method to solve such problems. Moreover, our heterogeneous-agent method allows the literature to avoid the approximation errors associated with discretizing heterogeneity when solving CDCPs (see Tintelnot 2017; Antràs et al. 2024b).

Our method allows models of multinational production to incorporate cross-location interdependencies and fixed costs without sacrificing tractability. Existing work avoids CDCPs by abstracting from these features (see Ramondo and Rodríguez-Clare 2013; Ramondo 2014; Arkolakis, Ramondo, et al. 2018); solves small CDCPs via brute force (see Zheng 2016; Tintelnot 2017; Dyrda, Hong, and Steinberg 2024); or restricts attention to positive complementarities to apply the method of Jia (2008) (see Antras, Fort, and Tintelnot 2017; Alfaro-Urena et al. 2023; Antràs et al. 2024b). We also show that complementarities and fixed costs are essential for quantifying the gains from multinational production, both in our calibrated model and by extending the sufficient-statistics framework of Arkolakis, Costinot, and Rodríguez-Clare (2012) to environments with these ingredients.

Finally, we also contribute to a growing literature on algorithmic solutions to CDCPs. Recent work imposes linear objective functions to enable integer programming (e.g., Head, Mayer, et al. 2024), adopts greedy algorithms that do not guarantee global optimality (e.g., Fan and Yang 2020), or uses deep learning to approximate policy functions in heterogeneous-agent CDCPs (e.g., Kulesza 2024). In contrast, our method applies to all objectives that satisfy our single crossing conditions, identifies the global optimum, and allows for exact aggregation across heterogeneous agents.

2. A Unified Framework to Solve CDCPs

In this section, we formally define combinatorial discrete choice problems and show how to solve them in cases where decisions satisfy a weak form of complementarity or substitutability. Throughout, L denotes a finite set of items indexed by ℓ; $\mathcal{P}(L)$ its power set, that is, the collection of all its possible subsets; and $\mathcal{L}$ an arbitrary element of the power set. We consider an objective function $f : \mathcal{P}(L) \to \mathbb{R}$ that maps elements from the power set to a real number and denote the space of such functions by $\mathcal{F} = \{f : \mathcal{P}(L) \to \mathbb{R}\}$. The Online Appendix contains a formal mathematical treatment of all statements and results.

2.1. Defining Combinatorial Discrete Choice Problems

Consider a decision-maker choosing a set $\mathcal{L} \in \mathcal{P}(L)$ to maximize an objective function f. Such discrete choice problems frequently appear in economic models, for example when a firm selects countries from which to import or a government chooses where to implement infrastructure projects. Our example throughout this section is a profit-maximizing firm selecting a set of foreign production locations $\mathcal{L}$.

Problems with multiple discrete choices are straightforward to solve when choices are independent, that is, when the valuation of any location $\ell \in L$ does not depend on the composition of the chosen set $\mathcal{L}$. In many settings, however, locations are interdependent: the value of a location ℓ depends on the other locations in $\mathcal{L}$. We introduce the following marginal value operator to formalize the notion of interdependencies among locations:

Definition 1 (Marginal Value Operator)**.** *For any $\ell \in L$ and $\mathcal{L} \in \mathcal{P}(L)$, define the marginal value operator $D_\ell : \mathcal{F} \to \mathcal{F}$ by*

$$D_\ell f(\mathcal{L}) \equiv f(\mathcal{L} \cup \{\ell\}) - f(\mathcal{L} \setminus \{\ell\}) .$$

If $D_\ell f(\mathcal{L})$ does not vary in $\mathcal{L}$ for each $\ell \in L$, the problem decomposes into $|L|$ independent decisions. If $D_\ell f(\mathcal{L})$ varies with $\mathcal{L}$, locations are interdependent and the firm must consider *combinations* of locations. We now formally define the class of combinatorial discrete choice problems.

Definition 2 (Combinatorial Discrete Choice Problem)**.** *A combinatorial discrete choice problem (CDCP) is the maximization problem*

$$\max_{\mathcal{L} \in \mathcal{P}(L)} f(\mathcal{L}) ,$$

where there is at least one $\ell \in L$ and two sets $\mathcal{L}, \mathcal{L}' \in \mathcal{P}(L)$ for which $D_\ell f(\mathcal{L}) \neq D_\ell f(\mathcal{L}')$.

We refer to these types of maximization problems as combinatorial because, generically, solving them requires evaluating the objective function for every element in the set $\mathcal{P}(L)$, which grows exponentially with the number of locations L.

2.2. Solving Combinatorial Discrete Choice Problems

Without additional structure on the objective function, no known polynomial-time algorithm exists for solving CDCPs, making the generic problem NP-hard. However, when the objective function f satisfies a single-crossing property, we show that an iterative solution method applies. In this section, we first introduce the property and then discuss its implications for solving CDCPs.

Single Crossing Differences in Choices The single crossing differences in choices property restricts how the marginal value of a location ℓ varies with the decision set $\mathcal{L}$.

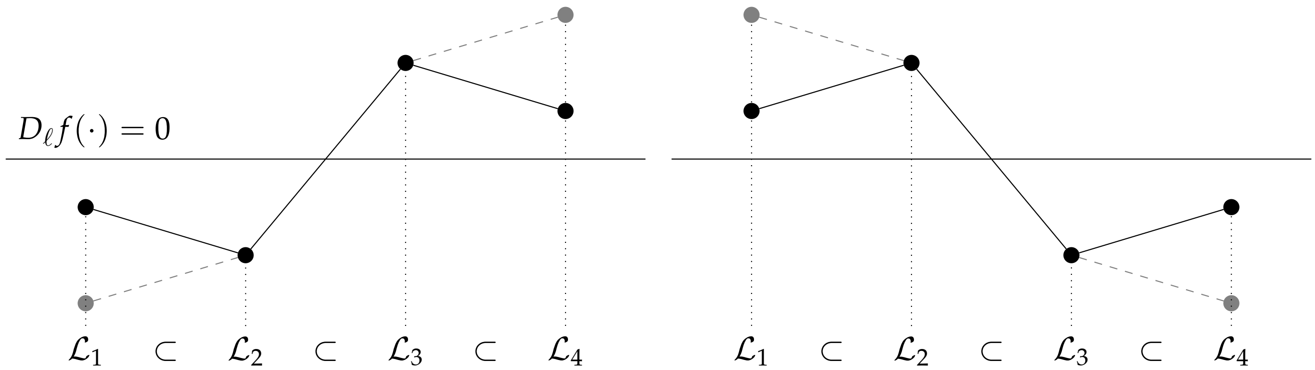
Definition 3 (SCD-C)**.** *The objective function f satisfies single crossing differences in choices from above if, for every $\ell \in L$, and for all decision sets $\mathcal{L}, \mathcal{L}' \in \mathcal{P}(L)$ such that $\mathcal{L} \subset \mathcal{L}'$,*

$$D_\ell f\left(\mathcal{L}'\right) \geq 0 \qquad \Rightarrow \qquad D_\ell f(\mathcal{L}) \geq 0 ,$$

and single crossing differences in choices from below if

$$D_\ell f(\mathcal{L}) \geq 0 \qquad \Rightarrow \qquad D_\ell f\left(\mathcal{L}'\right) \geq 0 .$$

The SCD-C condition restricts the marginal value function of location ℓ to change sign at most once. Thus, the conditions captures a weak form of complementarity, where a location



(A) SCD-C from below

(B) SCD-C from above

FIGURE 1: EXAMPLE MARGINAL VALUE FUNCTIONS

Both panels show the marginal value of location as a function of a succession of nested decision sets. The solid black line depicts a function that satisfies the weaker single crossing condition, but not the stricter super- or submodularity condition. The dashed line depicts a function that satisfies this stronger requirement. Note that under SCD-C, marginal values can increase or decrease arbitrarily as long as they only cross zero once.

with a positive marginal value for some decision set $\mathcal{L}$ retains a positive marginal value as additional locations are added (SCD-C from below) or removed (SCD-C from above) from $\mathcal{L}$. The solid and dashed lines in the left panel of Figure 1 both show examples of how the marginal value of a location ℓ can vary with SCD-C from below as locations are added to an initial location set $\mathcal{L}_1$. While the marginal value may change and even decrease (solid line), it crosses zero at most once from below. The right panel shows two marginal value functions with SCD-C from above, which can cross zero at most once from above.[1]

The well-known properties of supermodularity and submodularity capture a stronger notion of complementarity and serve as sufficient conditions for SCD-C to hold. A function f is submodular if marginal values are monotonically decreasing, that is if $D_\ell f(\mathcal{L}) \geq D_\ell f(\mathcal{L}')$ for all $\mathcal{L} \subseteq \mathcal{L}'$ and any ℓ, which implies SCD-C from above. Supermodularity reverses the inequality and is sufficient for SCD-C from below. While both the solid and dashed lines in Figure 1 show marginal value functions consistent with single crossing differences in choices, only the dashed lines are consistent with sub- or supermodularity.

The weaker form of complementarity implied by the SCD-C condition broadens the range of economic problems to which our methods apply. Consider a multinational firm like Ford, which profitably operates plants in Germany and the United States. Supermodularity requires

[1] Milgrom and Shannon (1994) introduce single-crossing conditions into economics to derive comparative statics in settings without differentiability. There is a terminological discrepancy, acknowledged in Milgrom (2004), between Milgrom and Shannon (1994) which uses "single crossing condition," and Milgrom (2004) which uses "single crossing differences." The condition we refer to as SCD-C is similar to, but weaker than, the quasi-supermodularity condition of Milgrom and Shannon (1994). We formally establish the relationship between these conditions in the Online Appendix. We adopt the "single-crossing *differences*" terminology, to emphasize that the *marginal value* of the choice changes sign at most once.

that adding a plant in Canada raises the marginal value of both the German and U.S. plants. In reality, however, the Canadian plant may lower the value of the U.S. plant—by cannibalizing its sales to the Canadian market—while increasing the value of the German plant through enhanced global scale economies. Although such a scenario violates super- and submodularity, it remains consistent with SCD-C from below, as long as the U.S. plant's marginal value stays weakly positive when the Canadian plant is added.

The Squeezing Procedure: Reducing a CDCP's Domain We now introduce a method for solving combinatorial discrete choice problems when the objective function satisfies either form of the SCD-C condition.

Our method iteratively shrinks the domain of the CDCP without eliminating the optimal decision set $\mathcal{L}^\star \equiv \arg\max_{\mathcal{L}\in\mathcal{P}(L)} f(L)$. To operationalize this approach, we introduce the notion of a "bounding pair," a pair of sets $[\underline{\mathcal{L}}, \overline{\mathcal{L}}]$ which defines a restricted subdomain of the original CDCP that always contains $\mathcal{L}^\star$, so that $\underline{\mathcal{L}} \subseteq \mathcal{L}^\star \subseteq \overline{\mathcal{L}}$. The full domain corresponds to the "trivial" bounding pair $[\emptyset, L]$. We refer to $\underline{\mathcal{L}}$ as the lower bounding set and $\overline{\mathcal{L}}$ as the upper bounding set, and say that $[\underline{\mathcal{K}}, \overline{\mathcal{K}}]$ is "tighter" than $[\underline{\mathcal{L}}, \overline{\mathcal{L}}]$ if $\underline{\mathcal{L}} \subseteq \underline{\mathcal{K}}$ and $\overline{\mathcal{K}} \subseteq \overline{\mathcal{L}}$.

We conceptualize solving a CDCP as eliminating all non-optimal decision sets from the problem's domain. We thus introduce a mapping with the goal of tightening a bounding pair around the optimal decision set $\mathcal{L}^\star$, progressively "squeezing" non-optimal sets out of the domain. We refer to this mapping as the squeezing step.

Definition 4 (Squeezing step). *The squeezing step is the mapping $S : \mathcal{P}(L) \times \mathcal{P}(L) \rightarrow \mathcal{P}(L) \times \mathcal{P}(L)$ defined for any $\mathcal{L}, \mathcal{L}' \in \mathcal{P}(L)$ as:*

$$S\left([\mathcal{L}, \mathcal{L}']\right) = \left[\inf\left\{\Phi(\mathcal{L}), \Phi\left(\mathcal{L}'\right)\right\}, \sup\left\{\Phi(\mathcal{L}), \Phi\left(\mathcal{L}'\right)\right\}\right],$$

where $\Phi : \mathcal{P}(L) \rightarrow \mathcal{P}(L)$ is defined as

$$\Phi(\mathcal{L}) = \{\ell \in L \mid D_\ell f(\mathcal{L}) > 0\}.$$

When the objective function satisfies SCD-C, iteratively applying the squeezing step to the trivial bounding pair $[\emptyset, L]$ generates a sequence of progressively tighter bounding pairs, converging in at most $|L|$ steps. We now provide a constructive proof of this claim before formally stating the result in Theorem 1.

The squeezing step builds on the mapping Φ, which identifies the set of locations $\ell \in L$ with positive marginal value given the decision set $\mathcal{L}$. Importantly, $\Phi(\mathcal{L}^\star) = \mathcal{L}^\star$ by construction, highlighting that Φ is the discrete analogue of a first order condition. Section 2.3 provides a detailed discussion of Jia (2008), which first introduced Φ in the context of a supermodular objective function.

The mapping Φ is monotone if and only if the underlying objective function f satisfies the SCD-C condition. The type of SCD-C determines the direction of the monotonicity. Consider two decision sets $\mathcal{L}, \mathcal{L}' \in \mathcal{P}(L)$ such that $\mathcal{L} \subset \mathcal{L}'$. With SCD-C from above, if a location ℓ has a positive marginal value in the larger set $\mathcal{L}$, it must also have a positive marginal value in any nested set $\mathcal{L}'$. As a result, $\mathcal{L} \subset \mathcal{L}'$ implies $\Phi(\mathcal{L}') \subseteq \Phi(\mathcal{L})$: the mapping Φ is order-reversing. With SCD-C from below, the logic is inverted: if ℓ has a positive marginal value as part of the smaller set $\mathcal{L}$, then it must have positive marginal value as part of the larger set. Then, $\mathcal{L} \subset \mathcal{L}'$ implies $\Phi(\mathcal{L}) \subseteq \Phi(\mathcal{L}')$ and the mapping Φ is order-preserving.[2]

The monotonicity of Φ simplifies the mapping S when we know the type of SCD-C the objective function f satisfies. With SCD-C from above, the mapping Φ is order-reversing, so that applying Φ to the lower bounding set must return a decision set that is always nested in the decision set that results from applying Φ to the upper bounding set. As a result, $S\left(\left[\underline{\mathcal{L}}, \overline{\mathcal{L}}\right]\right) = \left[\Phi\left(\overline{\mathcal{L}}\right), \Phi\left(\underline{\mathcal{L}}\right)\right]$. With SCD-C from below, the reverse logic simplifies the squeezing step to $S\left(\left[\underline{\mathcal{L}}, \overline{\mathcal{L}}\right]\right) = \left[\Phi\left(\underline{\mathcal{L}}\right), \Phi\left(\overline{\mathcal{L}}\right)\right]$.

The monotonicity of Φ also ensures that the pair of sets returned when applying the squeezing step to a bounding pair is itself always a valid bounding pair. In particular, consider a bounding pair $\left[\underline{\mathcal{L}}, \overline{\mathcal{L}}\right]$, so that $\underline{\mathcal{L}} \subseteq \mathcal{L}^\star \subseteq \overline{\mathcal{L}}$. If f satisfies SCD-C from above, applying Φ reverses this order, so $\Phi\left(\overline{\mathcal{L}}\right) \subseteq \Phi\left(\mathcal{L}^\star\right) \subseteq \Phi\left(\underline{\mathcal{L}}\right)$. Moreover, since $\Phi(\mathcal{L}^\star) = \mathcal{L}^\star$ by construction, this expression simplifies to $\Phi\left(\overline{\mathcal{L}}\right) \subseteq \mathcal{L}^\star \subseteq \Phi\left(\underline{\mathcal{L}}\right)$ so that $\left[\Phi\left(\overline{\mathcal{L}}\right), \Phi\left(\underline{\mathcal{L}}\right)\right]$ forms a new bounding pair. Similarly, with SCD-C from below, the order-preserving nature of Φ implies $\Phi\left(\underline{\mathcal{L}}\right) \subseteq \mathcal{L}^\star \subseteq \Phi\left(\overline{\mathcal{L}}\right)$.

Importantly, the monotonicity of Φ guarantees that, starting from the trivial bounding pair $[\emptyset, L]$, iteratively applying the squeezing step produces bounding pairs that weakly tighten with each iteration. For a concrete example, suppose f satisfies SCD-C from above, so that Φ is order-reversing. Then applying the squeezing step to the trivial bounding pair yields

$$\emptyset \subseteq \mathcal{L}^\star \subseteq L \qquad\qquad \Rightarrow \qquad\qquad \Phi(L) \subseteq \mathcal{L}^\star \subseteq \Phi(\emptyset) \ .$$

Note that the new bounding $[\Phi(L), \Phi(\emptyset)]$ is vacuously tighter than the trivial bounding pair $[\emptyset, L]$. Applying the squeezing step again produces:

$$\Phi(L) \subseteq \mathcal{L}^\star \subseteq \Phi(\emptyset) \qquad\qquad \Rightarrow \qquad\qquad \Phi(\Phi(\emptyset)) \subseteq \mathcal{L}^\star \subseteq \Phi(\Phi(L)) \ .$$

[2]We show the converse for the order-preserving case. Suppose Φ is order-preserving, and let $\mathcal{L} \subset \mathcal{L}'$ be arbitrary decision sets. Let ℓ be an arbitrary location. Now suppose $D_\ell f(\mathcal{L}) \geq 0$. Then, $\ell \in \Phi(\mathcal{L}) \subseteq \Phi(\mathcal{L}')$ since Φ is order-preserving; but $\ell \in \Phi(\mathcal{L}')$ implies $D_\ell f(\mathcal{L}') \geq 0$ by definition of Φ. As a result, Φ is order-preserving implies that $D_\ell f(\mathcal{L}) \geq 0 \Rightarrow D_\ell f(\mathcal{L}') \geq 0$. The proof in the order-reversing case is similar.

The order-reversing property of Φ guarantees that the new upper bounding set $\Phi(\Phi(L))$ is (weakly) tighter than the previous upper bounding set $\Phi(\varnothing)$: note $\varnothing \subseteq \Phi(L) \Rightarrow \Phi(\Phi(L)) \subseteq \Phi(\varnothing)$ and similarly, $\Phi(\varnothing) \subseteq L \Rightarrow \Phi(L) \subseteq \Phi(\Phi(\varnothing))$. Combining the two last steps, we conclude that

$$\varnothing \subseteq \Phi(L) \subseteq \Phi(\Phi(\varnothing)) \subseteq \mathcal{L}^\star \subseteq \Phi(\Phi(L)) \subseteq \Phi(\varnothing) \subseteq L\,.$$

Extending this logic inductively, each repeated application of the squeezing step produces a tighter bounding pair. Note that this procedure must converge in at most $|L|$ steps since, in each iteration, at least one location ℓ must be added to the lower bounding set or removed from the upper bounding set and there are $|L|$ total locations. A parallel argument, similarly appealing to the monotonicity of Φ, holds when the objective function satisfies SCD-C from below.

The following theorem formalizes the result.

Theorem 1. *If the objective function f exhibits SCD-C from either above or below, iteratively applying the squeezing step to the trivial bounding pair $[\varnothing, L]$ yields a sequence of sets*

$$\varnothing \subseteq \ldots \subseteq \underline{\mathcal{L}}^{(k)} \subseteq \underline{\mathcal{L}}^{(k+1)} \subseteq \mathcal{L}^\star \subseteq \overline{\mathcal{L}}^{(k+1)} \subseteq \overline{\mathcal{L}}^{(k)} \subseteq \ldots \subseteq L,$$

where k indexes the output of the kth application of the squeezing step. The squeezing step always converges, taking K applications where $K \leq |L|$.

We refer to the iterative application of the squeezing step until convergence as the *squeezing procedure*, and denote the corresponding operator by S^K, so that $\left[\underline{\mathcal{L}}^{(K)}, \overline{\mathcal{L}}^{(K)}\right] = S^K([\varnothing, L])$. If the converged lower and upper bounding sets coincide, then it must be that $\underline{\mathcal{L}}^{(K)} = \mathcal{L}^\star = \overline{\mathcal{L}}^{(K)}$, by definition of a bounding pair. If the two bounding sets are not identical, they nevertheless define a (weakly) smaller subdomain of the original CDCP, $\left[\underline{\mathcal{L}}^{(K)}, \overline{\mathcal{L}}^{(K)}\right]$, which we refer to as the *reduced* domain.

The Branching Procedure: Identifying the Optimal Decision Set We introduce a recursive branching procedure as a refinement method that identifies $\mathcal{L}^\star$ on the reduced domain. The branching step selects a location ℓ from $\overline{\mathcal{L}} \setminus \underline{\mathcal{L}}$, creates two "branches" defined by the bounding pairs $\left[\underline{\mathcal{L}} \cup \{\ell\}, \overline{\mathcal{L}}\right]$ and $\left[\underline{\mathcal{L}}, \overline{\mathcal{L}} \setminus \{\ell\}\right]$, then applies the squeezing procedure to each. Formally, we define the branching step as follows.

Definition 5 (Branching step)**.** *Given a bounding pair $\left[\underline{\mathcal{L}}, \overline{\mathcal{L}}\right]$ and element $\ell \in \overline{\mathcal{L}} \setminus \underline{\mathcal{L}}$, the branching*

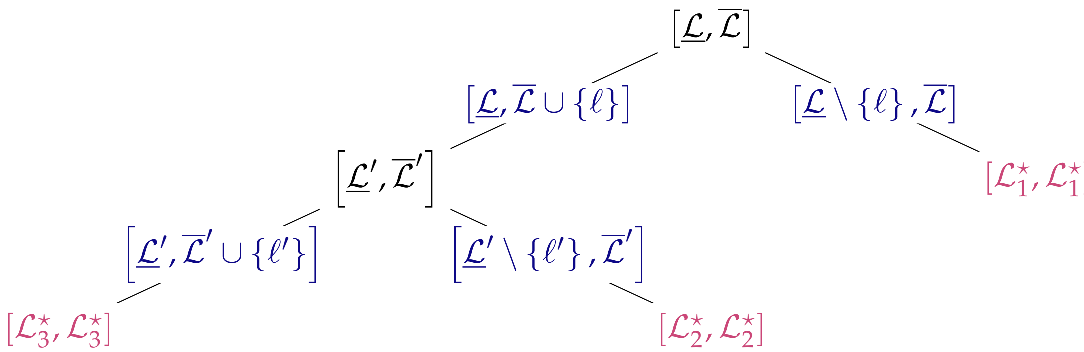


FIGURE 2: AN EXAMPLE PATH OF THE BRANCHING PROCEDURE

This figure shows an example of a tree of subproblems resulting from applying the branching procedure recursively. Convergence on a single branch occurs when the squeezing procedure returns a conditionally optimal set, indicated by a terminal node. The final output of the full recursive procedure is the collection of all conditionally optimal sets, in this example $\{\mathcal{L}_1^\star, \mathcal{L}_2^\star, \mathcal{L}_3^\star\}$.

step is defined as

$$B_\ell\left([\underline{\mathcal{L}}, \overline{\mathcal{L}}]\right) = \left\{S^K\left([\underline{\mathcal{L}} \cup \{\ell\}, \overline{\mathcal{L}}]\right), S^K\left([\underline{\mathcal{L}}, \overline{\mathcal{L}} \setminus \{\ell\}]\right)\right\}.$$

Figure 2 illustrates an example of the branching step's application: for a given converged bounding pair, $[\underline{\mathcal{L}}, \overline{\mathcal{L}}]$, that resulted from the squeezing procedure, two branches are created by applying the branching step with $\ell \in \overline{\mathcal{L}} \setminus \underline{\mathcal{L}}$. The branch on the right, which excludes ℓ, converges to a bounding pair with identical lower and upper bounds, identifying $\mathcal{L}_1^\star$ as the optimal decision set *conditional* on excluding ℓ. If squeezing does not converge to identical lower and upper bounds, as on the branch including ℓ on the left, branching recurs. A new location $\ell' \in \overline{\mathcal{L}}' \setminus \underline{\mathcal{L}}'$ is selected among those that remain, creating an additional branch. This recursive process builds a tree, terminating in a conditionally optimal decision set on each branch.

We refer to the recursive application of the branching step until global convergence as the *branching procedure* and define $\Lambda\left([\underline{\mathcal{L}}, \overline{\mathcal{L}}]\right)$ as the collection of conditionally optimal decision sets after global convergence. As we discuss in Section 2.3, the collection of decision sets returned by this recursive procedure turns out not to depend on which locations are selected for branching, so this definition is without ambiguity. The globally optimal decision set is the element in $\Lambda\left([\underline{\mathcal{L}}, \overline{\mathcal{L}}]\right)$ that yields the highest value of the objective function, so that $\mathcal{L}^\star = \arg\max_{\mathcal{L} \in \Lambda([\underline{\mathcal{L}}, \overline{\mathcal{L}}])} f(\mathcal{L})$.

The incumbent alternative to branching is to use the "brute force" method of evaluating the objective function at every element in the reduced domain produced by the squeezing procedure. Intuitively, the branching procedure applies the squeezing procedure as much as possible, yielding to brute force only one location ℓ at a time. As a result, branching is typically faster than brute force. In the worst case, the squeezing step eliminates no decision set on any branch and hence reduces to evaluating the objective function at all elements in the reduced domain just like brute force. In the Online Appendix, we show that this worst case never occurs when the objective function satisfies SCD-C from above.

2.3. The Mathematics of Squeezing and Branching

This section explains the mathematical foundation of our squeezing method and formally connects it to Jia (2008).

Jia (2008), Supermodularity, and SCD-C from Below Jia (2008) introduced the Φ mapping to the economics literature, in the context of a chain store that chooses a set of store locations. The paper recasts finding the optimal decision set of store locations, $\mathcal{L}^\star$, as finding the fixed points of Φ, since $\Phi(\mathcal{L}^\star) = \mathcal{L}^\star$ by construction.[3]

Crucially, Jia (2008) shows that the mapping Φ is order-preserving when the underlying objective function is supermodular, as in the chain store application of the paper. When Φ is order-preserving, the theorem of Tarski (1955) guarantees the existence of well-defined smallest and largest fixed points, $\mathcal{L}^{\inf}$ and $\mathcal{L}^{\sup}$, which together form a natural bounding pair $\left[\mathcal{L}^{\inf}, \mathcal{L}^{\sup}\right]$, since necessarily $\mathcal{L}^{\inf} \subseteq \mathcal{L}^\star \subseteq \mathcal{L}^{\sup}$.

The method of Jia (2008) to identify the bounding pair $\left[\mathcal{L}^{\inf}, \mathcal{L}^{\sup}\right]$ is rooted in Kleene's fixed point theorem, which states that iteratively applying an order-preserving map to $\emptyset$ always converges to its smallest fixed point $\mathcal{L}^{\inf}$, while applying it to L always converges to its largest fixed point $\mathcal{L}^{\sup}$.[4] In cases where the bounding pair $\left[\mathcal{L}^{\inf}, \mathcal{L}^{\sup}\right]$ does not identify $\mathcal{L}^\star$, Jia (2008) applies a brute force search on the reduced domain.

In the case of SCD-C from below, our squeezing procedure implements the same method as Jia (2008). With an order-preserving Φ, the squeezing step simplifies to $S\left(\left[\underline{\mathcal{L}}, \overline{\mathcal{L}}\right]\right) = \left[\Phi\left(\underline{\mathcal{L}}\right), \Phi\left(\overline{\mathcal{L}}\right)\right]$, which is equivalent to applying the Φ mapping separately to the lower and upper bounding sets. Consequently, the squeezing procedure produces a converged bounding pair that coincides with the smallest and largest fixed points of Φ, so that $S^K\left([\emptyset, L]\right) = \left[\underline{\mathcal{L}}^{(K)}, \overline{\mathcal{L}}^{(K)}\right] = \left[\mathcal{L}^{\inf}, \mathcal{L}^{\sup}\right]$.

Our first contribution relative to Jia (2008) is hence to extend the paper's method beyond the context of supermodular objective functions by showing that SCD-C from below is the

[3] The mapping Φ has antecedents in the Operations Research literature on Boolean optimization. It presents a simple updating rule for decisions based on the discrete analogue of a first order condition (see, e.g., Boros and Hammer 2002).

[4] See Stoltenberg-Hansen, Lindström, and Griffor (1994) for an exposition. The fixed-point theorem derives from results first established in Kleene (1936) and Kleene (1938)—though it was not stated explicitly there—and is named accordingly.

necessary and sufficient condition for Φ to be order-preserving.

The Challenges of SCD-C from Above and Fixed Edges Our second contribution is to develop a solution method that applies when the objective function satisfies SCD-C from above. The key difficulty in this case is that Φ is order-reversing so that the fixed point theorem of Tarski (1955), and Kleene's by extension, does not apply; a smallest and largest fixed point bounding the optimal decision set may not exist.

This limitation reflects a basic economic feature of negative complementarities. Consider a firm choosing between two perfectly substitutable production locations, $L = \{\text{USA}, \text{CAN}\}$. If no other location is active, either has positive marginal value. However, when both are active, neither does, as each fully substitutes for the other. Applying the algorithm in Jia (2008) yields $\Phi(\emptyset) = \{\text{USA}, \text{CAN}\}$ and $\Phi(\{\text{USA}, \text{CAN}\}) = \emptyset$, so iterating from either extreme oscillates perpetually.

Though repeated application of Φ may not converge to a fixed point with SCD-C from above, it does result in a different type of convergence which alternates between two sets, as in the previous example. We refer to such a pair of sets $[\mathcal{L}, \mathcal{L}']$ for which $\mathcal{L} = \Phi(\mathcal{L}')$ and $\mathcal{L}' = \Phi(\mathcal{L})$ as a "fixed edge" of the Φ mapping. The concept of a fixed edge is a strict generalization of the concept of a fixed point, since if $\mathcal{L}$ is a fixed point of Φ, then the "pair" $[\mathcal{L}, \mathcal{L}]$ is a fixed edge.

Klimeš (1981) shows that, though an order-reversing map need not have a smallest and largest fixed point, it always has a fixed edge $\left[\mathcal{L}^{\inf}, \mathcal{L}^{\sup}\right]$ that is "extreme" in the sense that $\mathcal{L}^{\inf} \subseteq \mathcal{L} \subseteq \mathcal{L}' \subseteq \mathcal{L}^{\sup}$ holds for all other fixed edges $[\mathcal{L}, \mathcal{L}']$.[5] As a result, the extreme fixed edge serves as a bounding pair in the same way the smallest and largest fixed point do with Tarski (1955). With SCD-C from above, the squeezing procedure applied to the trivial bounding pair converges to the extreme fixed edge of Φ.

The insights in this section imply that the squeezing step S is itself an increasing mapping whenever the underlying objective function satisfies either of the SCD-C conditions. As a result, an alternative proof for Theorem 1 follows by applying the theorem of Tarski (1955) and Kleene's theorem to S directly.

The above discussion shows that the squeezing procedure converges to $[\mathcal{L}^\star, \mathcal{L}^\star]$ if and only if $\mathcal{L}^\star$ is the *unique* fixed point of Φ. On the other hand, the branching procedure collects all the fixed points of Φ. In particular, the set of conditionally optimal decision sets at the terminal node of each branch, $\Lambda\left(\underline{\mathcal{L}}, \overline{\mathcal{L}}\right)$, is the collection of all the fixed points of Φ. By implication, the branching procedure results in the same set of conditionally optimal decision sets, regardless of which items are selected for branching, and collapses to brute force only in the extreme case

[5] The intuition behind Klimeš (1981) is simple. Suppose Φ is order-reversing. Then $g \equiv \Phi \circ \Phi$ is order-preserving. By Tarski (1955), g has a set of fixed points with a smallest and largest element. Any fixed point of g must be such that $\Phi(\Phi(\mathcal{L})) = \mathcal{L}$. Then, the pair $[\mathcal{L}, \Phi(\mathcal{L})]$ together must be a fixed edge of Φ.

where every set in the reduced domain returned by squeezing is a fixed point of Φ.

2.4. Solving CDCPs for Heterogeneous Agents

We now extend the combinatorial discrete choice framework to a setting with heterogeneous agents. We consider an objective function $f : \mathcal{P}(L) \times \mathbb{R} \to \mathbb{R}$, that maps a set of items $\mathcal{L}$ and an agent type $z \in \mathbb{R}$ to a scalar payoff $f(\mathcal{L}, z)$. Our exposition focuses on single-dimensional heterogeneity, though our results extend to multidimensional settings (see Arkolakis, Eckert, and Shi 2023).

In the context of heterogeneous agents, the object of interest is the "policy function" that maps an agent's type to its optimal decision set, encoding each agent's solution to the CDCP.

Definition 6 (Policy Function)**.** *The policy function $\mathcal{L}^\star : \mathbb{R} \to \mathcal{P}(L)$ specifies the optimal decision set for each type z, so that $\mathcal{L}^\star(z) = \arg\max_{\mathcal{L} \in \mathcal{P}(L)} f(\mathcal{L}, z)$.*

In the multinational production setting that serves as our example, the policy function maps the productivity of a firm to its optimal combination of production locations. This policy function plays a central role in aggregating firm-level decisions to solve the general equilibrium of the model. For example, computing the aggregate price index requires integrating the policy function over the full support of the firm productivity distribution. We now present an iterative method to recover the policy function for objective functions that satisfy SCD-C and an additional single crossing differences in type condition.

Single Crossing Differences in Type The single crossing differences in type property restricts how a firm's productivity affects the marginal value of a location.

Definition 7 (SCD-T)**.** *The objective function f exhibits single crossing differences in type if, for all items $\ell \in L$, decision sets $\mathcal{L} \in \mathcal{P}(L)$, and types $z, z' \in \mathbb{R}$ such that $z < z'$:*

$$D_\ell f(\mathcal{L}, z) \geq 0 \qquad \Rightarrow \qquad D_\ell f(\mathcal{L}, z') \geq 0\,.$$

Intuitively, the SCD-T condition ensures that if, as part of a decision set $\mathcal{L}$, a location ℓ has a positive marginal value for a firm of productivity z, then it must also have a positive marginal value for any firm of higher productivity $z' > z$. If instead the marginal value of ℓ crosses zero from above, the problem can be reformulated using the transformation $\tilde{z} \equiv -z$ to satisfy SCD-T.[6]

[6] This property corresponds exactly to the single-crossing condition introduced by Milgrom and Shannon (1994), later referred to as the single-crossing *differences* condition by Milgrom (2004). The firm type z can represent any characteristic, endogenous or exogenous, that affects payoffs, and the policy function $\mathcal{L}^\star(z)$ describes how optimal choices respond to changes in that characteristic. When z is endogenous, the policy function is often referred to as a best-response function.

The squeezing procedure uses the choice monotonicity implied by the SCD-C assumption to rule out many decision sets without explicitly evaluating their payoff. In a similar way, by appealing to the type monotonicity afforded by the SCD-T assumption, our method applies the choice monotonicity implied by the SCD-C restriction to discard many decision sets for entire ranges of productivity without evaluating the objective at any of these productivities. This approach is possible because, when the objective function satisfies both SCD-C and SCD-T, the policy function changes its value only at a finite number of cutoff productivities. As a result, instead of solving a CDCP for every productivity, it suffices to identify the cutoff productivities and the constant value of the policy function in between cutoffs.

When f satisfies supermodularity and SCD-T, Milgrom and Shannon (1994) show that the policy function exhibits a nesting structure: higher productivity firms choose all locations selected by lower productivity firms and possibly more, so that $z < z'$ implies $\mathcal{L}^{\star}(z) \subseteq \mathcal{L}^{\star}(z')$.[7] However, SCD-C and SCD-T alone are not sufficient to ensure nesting.

Solving for the Policy Function with SCD-C and SCD-T To solve for the policy function, we introduce a generalized squeezing procedure. As a first step, we extend the notion of the bounding pair $\left[\underline{\mathcal{L}}, \overline{\mathcal{L}}\right]$, associated with the CDCP of a single firm, to set-valued functions defined over productivities, $\underline{\mathcal{L}}(\cdot)$ and $\overline{\mathcal{L}}(\cdot)$. These "bounding set functions" are such that $\underline{\mathcal{L}}(z) \subseteq \mathcal{L}^{\star}(z) \subseteq \overline{\mathcal{L}}(z)$ for any productivity $z \in \mathbb{R}$. Our solution method iteratively tightens the bounding set functions around the policy function. We refer to $\left[\underline{\mathcal{L}}(\cdot), \overline{\mathcal{L}}(\cdot)\right] \equiv [\emptyset, L]$ as the trivial bounding set functions since these constant functions always nest the policy function for all z.

Any pair of bounding set functions implies a partition of productivities into intervals with *identical* lower and upper bounding sets. We define the partition $\mathcal{T}$ created by a pair of bounding set functions $\left[\underline{\mathcal{L}}(\cdot), \overline{\mathcal{L}}(\cdot)\right]$ as follows:

$$\mathcal{T}\left(\left[\underline{\mathcal{L}}(\cdot), \overline{\mathcal{L}}(\cdot)\right]\right) = \left\{\mathcal{Z}_1, \ldots \mathcal{Z}_t, \ldots \mathcal{Z}_T\right\}$$
$$\text{such that } \mathcal{Z}_t = \left\{z \in \mathbb{R} \mid \underline{\mathcal{L}}(z) = \underline{\mathcal{L}}_t, \overline{\mathcal{L}}(z) = \overline{\mathcal{L}}_t\right\},$$

where t indexes the productivity intervals for which the bounding pair functions imply identical bounding pairs. For brevity, we use $\mathcal{T}$ to denote the partition when the pair of bounding set functions is unambiguous.

Figure 3 illustrates how a pair of bounding set functions partitions the productivity types

[7] More precisely, Milgrom and Shannon (1994) show that nesting holds under the weaker condition of *quasi*-supermodularity, which is stronger than SCD-C as we discuss in the Online Appendix. This result has been widely used in applied theory and empirical work, including Antras, Fort, and Tintelnot (2017) in their analysis of firm-level sourcing decisions. It is also reminiscent of the positive assortative decision patterns studied in Costinot (2009).

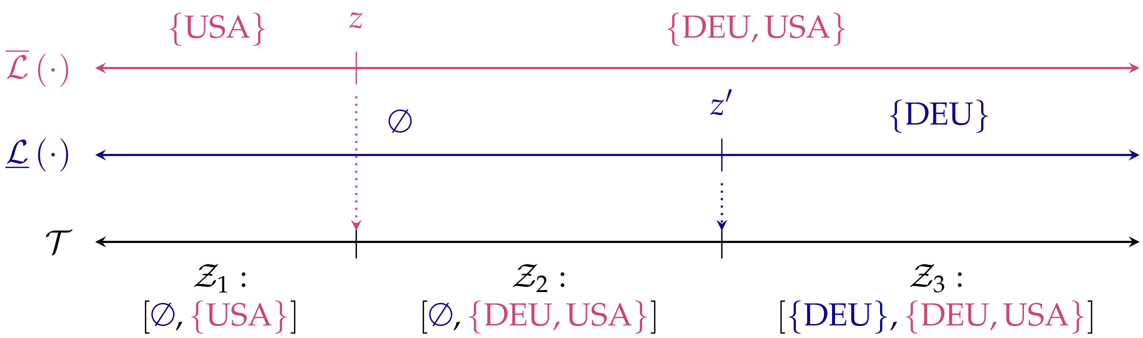


FIGURE 3: PARTITIONING PRODUCTIVITIES BY COMMON BOUNDING PAIRS

The top line illustrates an example upper bounding set function, while the middle illustrates an example lower bounding set function. Together, these two set-valued functions imply the partitioning $\mathcal{T}$, which creates intervals of productivity. In this figure, there are three intervals, so $\mathcal{T} = \{\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3\}$. All productivities within a interval $\mathcal{Z}_t$ share the listed bounding pair.

into intervals. The two lines at the top represent the lower and upper bounding set functions, each changing their values at different cutoffs, z and z'. The line at the bottom shows the resulting partition comprised of productivity ranges for which both bounding set functions are constant. These intervals identify the cutoff productivities at which either the lower or upper bound changes, and thus determine the set of relevant cutoffs for the policy function.

We now define a generalized version of the squeezing step above.

Definition 8 (Generalized squeezing step)**.** *Let $\mathcal{C} = \{c : \mathbb{R} \to \mathcal{P}(L)\}$ be the space of functions mapping types to decision sets. The generalized squeezing step is the mapping $S^g : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ defined for any $\mathcal{L}(\cdot), \mathcal{L}'(\cdot) \in \mathcal{C}$ as:*

$$S^g\left(\left[\mathcal{L}(\cdot), \mathcal{L}'(\cdot)\right]\right) = \Big[\inf\left\{\Phi^g\left(\mathcal{L}(\cdot), \cdot\right), \Phi^g\left(\mathcal{L}'(\cdot), \cdot\right)\right\}, \\ \sup\left\{\Phi^g\left(\mathcal{L}(\cdot), \cdot\right), \Phi^g\left(\mathcal{L}'(\cdot), \cdot\right)\right\}\Big]$$

where the mapping $\Phi^g : \mathcal{P}(L) \times \mathbb{R} \to \mathcal{P}(L)$ is defined as

$$\Phi^g(\mathcal{L}, z) = \left\{\ell \mid z \geq z^g_\ell(\mathcal{L})\right\}$$

and the functions $z^g_\ell : \mathcal{P}(L) \to \mathbb{R}$ are defined as $z^g_\ell(\mathcal{L}) = \inf\{z \mid D_\ell(\mathcal{L}, z) = 0\}$ for each $\ell \in L$.[8]

When the underlying objective function satisfies SCD-C and SCD-T, the generalized squeezing step applies the logic of the squeezing step from the single-firm case to the full range

[8] If $D_\ell(\mathcal{L}, z) < 0$ for all types z, we define $z^g_\ell(\mathcal{L}) \equiv \infty$. Likewise, if $D_\ell(\mathcal{L}, z) > 0$ for all types, $z^g_\ell(\mathcal{L}) \equiv -\infty$. Thus, formally, the range of each z^g_ℓ is the two-point compactification of the real line $\mathbb{R} \cup \{-\infty, \infty\}$.

of productivities. In particular, for each productivity value z, $\Phi^g(\mathcal{L}, z)$ coincides with $\Phi(\mathcal{L})$ when the objective function is evaluated at z.

The importance of the SCD-T assumption is that it eliminates the need to evaluate marginal values separately for each productivity level. Instead, for each location ℓ and decision set $\mathcal{L}$, there exists a *unique* productivity cutoff $z_\ell^g(\mathcal{L})$ that identifies the firm indifferent to including ℓ into $\mathcal{L}$. Rather than computing the marginal value of ℓ at each z, it suffices to check whether a firm's productivity lies above or below the cutoff $z_\ell^g(\mathcal{L})$.

We illustrate how the generalized squeezing step proceeds in practice for an objective function that satisfies SCD-C from above and SCD-T. Consider a pair of bounding set functions and the associated productivity partition $\mathcal{T}$. For a given interval $\mathcal{Z}_t \in \mathcal{T}$, we compute the two cutoffs $z_\ell^g(\underline{\mathcal{L}}_t)$ and $z_\ell^g(\overline{\mathcal{L}}_t)$ for each ℓ, where the SCD-C and SCD-T conditions together imply $z_\ell^g(\underline{\mathcal{L}}_t) \leq z_\ell^g(\overline{\mathcal{L}}_t)$. Then, for all firms with productivity $z < z_\ell^g(\underline{\mathcal{L}}_t)$ in $\mathcal{Z}_t$, ℓ is not part of the optimal decision set, so the upper bounding set function updates to $\overline{\mathcal{L}}_t \setminus \{\ell\}$ for these productivities. Conversely, for all firms with productivity $z_\ell^g(\overline{\mathcal{L}}_t) < z$ in $\mathcal{Z}_t$, ℓ is part of the decision set, so the lower bounding set function updates to $\underline{\mathcal{L}}_t \cup \{\ell\}$ for these productivities. Figure 3 depicts the outcome of updating the trivial bounding set functions where z corresponds to $z_\ell^g(\underline{\mathcal{L}}_t)$, z' corresponds to $z_\ell^g(\overline{\mathcal{L}}_t)$, and ℓ corresponds to DEU. A full application of the generalized squeezing step requires computing $2 \times |L|$ cutoffs for each interval $\mathcal{Z}_t$.

When the underlying objective function satisfies both SCD-C and SCD-T, applying the generalized squeezing step to a pair of bounding set functions always returns a (weakly) tighter pair of bounding set functions. Iteration of the generalized squeezing step converges once $S^g([\underline{\mathcal{L}}(\cdot), \overline{\mathcal{L}}(\cdot)]) = [\underline{\mathcal{L}}(\cdot), \overline{\mathcal{L}}(\cdot)]$. Since each bounding set can tighten at most $|L|$ times, the procedure converges in at most $|L|$ applications. The following theorem formalizes this result.

Theorem 2. *If the objective function f exhibits SCD-C and SCD-T, iteratively applying the generalized squeezing step to the trivial bounding set functions $\left[\underline{\mathcal{L}}^{(0)}(\cdot), \overline{\mathcal{L}}^{(0)}(\cdot)\right] = [\emptyset, L]$ yields a sequence of bounding set functions so that for all $z \in \mathbb{R}$,*

$$\emptyset \subseteq \ldots \subseteq \underline{\mathcal{L}}^{(k)}(z) \subseteq \underline{\mathcal{L}}^{(k+1)}(z) \subseteq \mathcal{L}^\star(z) \subseteq \overline{\mathcal{L}}^{(k+1)}(z) \subseteq \overline{\mathcal{L}}^{(k)}(z) \subseteq \ldots \subseteq L,$$

where k indexes the output of the kth application of the generalized squeezing step. The generalized squeezing step always converges, taking K' applications where $K' \leq |L|$.

At convergence, the optimal decision set is identified for any interval $\mathcal{Z}_t^{(K')} \in \mathcal{T}^{(K')}$ where the bounds coincide, that is, $\overline{\mathcal{L}}_t^{(K')} = \underline{\mathcal{L}}_t^{(K')} = \mathcal{L}_t^\star$. In intervals where the bounds differ, the generalized squeezing step has converged without identifying the optimal decision set. For such cases, we define two refinement methods that always converge to the policy function, including a *generalized branching* procedure, in the Online Appendix.

In what follows, we refer to the application until convergence of generalized squeezing and its refinement as the "policy function method." In contrast to discretization or bisection methods, the policy function method uses the monotonicity afforded by the SCD-T restriction to identify the exact cutoffs at which the policy function changes its value. This approach eliminates unnecessary computation at non-cutoff productivities. By recovering the exact policy function, we also avoid the interpolation between productivities required by discretization methods. Section 5.1 shows that such interpolation can introduce substantial errors in aggregate variables in the context of a realistically calibrated model of multinational production.

3. A Quantitative Model of Multinational Production

In this section, we introduce a quantitative model of multinational production (MP). We characterize the firm's location problem and the economic mechanisms that turn it into a CDCP, then derive explicit parameter conditions under which SCD-C and SCD-T hold. In the Online Appendix, we extend the framework to accommodate a broader class of demand and cost functions.

Setup The world economy consists of a discrete set of countries L. We index firm headquarter locations by i, production locations by ℓ, and final consumption locations by n. Each firm produces a differentiated final good variety ω. Every production location ℓ has a mass of households H_ℓ, each of which inelastically supply one unit of labor at wage w_ℓ. Labor markets are perfectly competitive and output markets are monopolistically competitive.

Demand System Consumers in all destinations have identical CES preferences with elasticity σ over the set of available final goods. Then, the demand function for good ω as a function of its destination-specific price $p_n(\omega)$ is

$$q_n\left(p_n\left(\omega\right)\right) = Q_n\left(\frac{p_n\left(\omega\right)}{P_n}\right)^{-\sigma},$$

where Q_n is the CES consumption basket and P_n its ideal price index. Firms and consumers take the aggregate objects Q_n and P_n as given.

Production Technology Consider a firm headquartered in location i that produces the final good ω and operates a set of production locations $\mathcal{L} \subseteq L$. The firm has productivity $z(\omega)$, which describes its efficiency of producing good ω in any potential location ℓ. We index firms by i and z alone, anticipating that firms with identical headquarter location and productivity behave identically.

The firm delivers its final good to destination n by combining production from its locations ℓ. The resulting marginal cost of a firm headquartered in location i of delivering a unit of its output to destination n is given by a constant-elasticity aggregator over the marginal cost of each of its active production locations:

$$c_{in}\left(\mathcal{L},z\right)=\frac{1}{z}\left[\sum_{\ell\in\mathcal{L}}\xi_{i\ell n}^{-\theta}\right]^{-\frac{1}{\theta}} \qquad \text{where} \qquad \xi_{i\ell n}=\gamma_{i\ell}\frac{w_\ell}{T_\ell}\tau_{\ell n}. \tag{1}$$

For each production location, w_ℓ is the equilibrium wage rate and T_ℓ is an exogenous location-specific productivity shifter common to all firms producing there. Firms also face a bilateral cost of multinational production $\gamma_{i\ell}$, or MP cost, which captures factors such as communication or language costs, as well as a bilateral cost of trade $\tau_{\ell n}$. We summarize all cost shifters in a trilateral resistance term denoted $\xi_{i\ell n}$.

All else equal, the marginal cost in equation (1) increases in wages, MP costs, and trade costs, while it decreases in firm and country productivity. Crucially, the marginal cost always declines as the production location set $\mathcal{L}$ grows: a firm that operates an expanded set of production sites has lower marginal cost of supplying its good to any final destination.

The elasticity of substitution across locations, $\theta > 0$, captures that the firm may require all of its locations for production. In the limiting case as $\theta \to \infty$, the firm uses only its lowest-cost location.

The elasticity θ determines whether there are increasing, constant, or decreasing returns to additional locations on the cost side. For $\theta > 1$, locations are substitutes so that, all else equal, any additional production location lowers marginal costs by less; on the other hand, with $0 < \theta < 1$, locations are complements so that any additional production location lowers marginal costs by more.

Several microfoundations can give rise to the cost aggregator in equation (1). In Appendix A, we present one based on Fréchet-distributed cost shocks, closely aligned with the frameworks in Antras, Fort, and Tintelnot (2017) and Tintelnot (2017). In Antras, Fort, and Tintelnot (2017), final-good firms do not operate sourcing locations themselves or export; instead, they pay fixed costs to add sourcing partners, each subject to Fréchet-distributed cost shock draws. The setup in Tintelnot (2017), which is closer to ours, features firms that produce and export bundles of goods and pay fixed costs to establish plants across locations. The marginal cost of producing each good at a plant depends on plant-specific Fréchet shocks. While these models differ in structure, they ultimately deliver the same cost aggregator. Importantly, conditional on equation (1), the equilibrium of our model implies the same aggregate allocations regardless of the specific microfoundation.[9]

Profit Maximization The profit maximization problem of the firm has two parts: choosing a set of production locations $\mathcal{L}$ and setting the price of its final good in each destination market. Firms set the price for their final good in destination n as a constant markup over their marginal cost, $c_{in}(\mathcal{L}, z)$, so that

$$p_{in}(\mathcal{L}, z) = \frac{\sigma}{\sigma - 1} c_{in}(\mathcal{L}, z) .$$

Finally, we turn to the choice of the production location set $\mathcal{L}$. To produce in location ℓ, a firm headquartered in i must pay a fixed cost $f_{i\ell}$, denominated in location ℓ production labor. Without the fixed cost $f_{i\ell}$, all firms would produce in all locations ℓ and there would be no (combinatorial) discrete choice problem to solve.

The firm headquartered in location i with productivity z chooses the optimal location decision set $\mathcal{L}^{\star}(z)$ to maximize its operating profits:

$$\max_{\mathcal{L} \in \mathcal{P}(L)} \pi_i(\mathcal{L}, z) \equiv \max_{\mathcal{L} \in \mathcal{P}(L)} \left\{ \sum_n \frac{1}{\sigma - 1} q_{in}(\mathcal{L}, z) c_{in}(\mathcal{L}, z) - \sum_{\ell \in \mathcal{L}} w_\ell f_{i\ell} \right\} , \tag{2}$$

where $q_{in}(\mathcal{L}, z) \equiv q_n(p_{in}(\mathcal{L}, z))$ is the demand for the final good in destination market n.

To establish a headquarter in location i, firms first must pay a labor-denominated entry cost f_i^e to draw a productivity z from an exogenous distribution $G_i(z)$. Given their productivity z, firms with non-negative operating profits produce positive quantities, while firms with negative operating profits shut down right after paying f_i^e.

Aggregation and the Equilibrium System We now turn to aggregation over firms and the determination of aggregate variables in general equilibrium.

The first equilibrium condition is a zero profit condition that pins down the cutoff productivity $\tilde{z}_i$ below which firms would have negative global operating profits and thus exit instead:

$$\pi_i(\mathcal{L}^{\star}(\tilde{z}_i), \tilde{z}_i) = 0 . \tag{3}$$

The second equilibrium condition is a free entry condition that reflects that firms enter until their expected operating profits before drawing a productivity are zero. The free entry

[9] In particular, Tintelnot (2017) assumes that the elasticity of substitution within and across firms is identical, which requires the Fréchet shape parameter to exceed this common elasticity, restricting the problem to the SCD-C from above case as we discuss below. By allowing the two elasticities to differ, as in our setup, this restriction is relaxed. The Fréchet distribution can also be replaced by a multivariate Pareto, as in Arkolakis, Ramondo, et al. (2018). Alternatively, one can microfound the cost aggregator by assuming that firms produce intermediate goods in each location that are incomplete substitutes, in which case the Fréchet parameter is replaced by the elasticity of substitution across intermediates, as in Antràs et al. (2024a).

condition pins down the total mass of entrants M_i in each origin country i by equalizing the entry cost on the left with the operating profits of the average firm across all destination markets on the right:

$$w_i f_i^e = \int_{\tilde{z}_i}^{\infty} \pi_i \left(\mathcal{L}^\star \left(\tilde{z}_i \right), \tilde{z}_i \right) \mathrm{d}G_i(z). \tag{4}$$

Price indices in each destination market n aggregate the individual prices of all goods offered:

$$P_n^{1-\sigma} = \sum_i M_i \int_{\tilde{z}_i}^{\infty} p_{in} \left(\mathcal{L}_i^\star \left(z \right), z \right)^{1-\sigma} \mathrm{d}G_i \left(z \right). \tag{5}$$

In addition, the labor market must clear in each production location ℓ. There are three sources of labor demand: variable labor requirements from all the production sites operating in country ℓ, the entry costs of all the firms headquartered in ℓ, and the fixed costs incurred by foreign and domestic firms to set up production in location ℓ. Labor market clearing equates the total labor supply in location ℓ to total labour demand as follows:

$$\begin{aligned} w_\ell H_\ell = {} & \frac{\sigma - 1}{\sigma} \sum_{i,n} M_i X_n \int_{\tilde{z}_i}^{\infty} \frac{\mathbb{1}_{i\ell} \left(z \right) \xi_{i\ell n}^{-\theta}}{\sum_{k \in \mathcal{L}_i^\star (z)} \xi_{ikn}^{-\theta}} \left[\frac{p_{in} \left(\mathcal{L}_i^\star \left(z \right), z \right)}{P_n} \right]^{1-\sigma} \mathrm{d}G_i \left(z \right) \\ & + M_\ell w_\ell f_\ell^e + \sum_i M_i w_\ell f_{i\ell} \int_{\tilde{z}_i}^{\infty} \mathbb{1}_{i\ell} \left(z \right) \mathrm{d}G_i \left(z \right), \end{aligned} \tag{6}$$

where the indicator $\mathbb{1}_{i\ell} \left(z \right) \equiv \mathbb{1} \left\{ \ell \in \mathcal{L}_i^\star \left(z \right) \right\}$ takes the value 1 if a firm with productivity z from origin i opens a production location in country ℓ.[10] The term X_n denotes total expenditure on final goods in destination n.

Lastly, balance of payments requires that total expenditure from consumers in a market n equals their total income:

$$X_n = w_n H_n. \tag{7}$$

The general equilibrium in our model is defined as follows.

Definition. General equilibrium is a set of policy functions $\left\{ \mathcal{L}_i \left(\cdot \right) \right\}_i$ and aggregate variables $\left\{ w_i, \tilde{z}_i, M_i, P_i, X_i \right\}_i$ so that

1. given the aggregate variables, the policy functions solve firms' optimization problems in equation (2); and
2. given the policy functions, the aggregate variables satisfy equations (3)–(7).

Before describing how to quantify the model using data on trade and MP flows, we first describe how to establish the SCD-C and SCD-T properties in our framework.

[10] Note that under CES demand, a fraction $(\sigma - 1)/\sigma$ of a firm's total sales are variable production costs; location $\ell \in \mathcal{L}_i^\star \left(z \right)$ accounts for a share $\mathbb{1}_{i\ell} \left(z \right) \xi_{i\ell n}^{-\theta} / \sum_{k \in \mathcal{L}_i^\star (z)} \xi_{ikn}^{-\theta}$ of these costs.

Establishing the SCD-C and SCD-T conditions Our model features two forces that generate interdependencies among the production locations of an individual firm. On the demand side, the demand elasticity σ indexes the firm's ability to scale. The larger is σ, the more sensitive demand is to prices, and hence the more the marginal cost savings of large multinationals translates into larger sales. All else equal, such returns to scale make each additional location more valuable, generating positive complementarities among locations. On the cost side, when $\theta > 1$, production locations are substitutes as they compete with one another to supply any destination ("cannibalization"). However, if $0 < \theta < 1$, locations act as complements in production. Overall, when $\theta > 1$, the cost- and demand-side effects oppose each other; when $\theta < 1$, they reinforce each other.

The model generates a closed-form condition on parameters that determines the type of complementarities among locations in the firm's CDCP. As we show formally in the Online Appendix, negative complementarities dominate and the firm's profit function satisfies SCD-C from above iff $\sigma - 1 < \theta$, whereas positive complementarities prevail and SCD-C from below holds iff $\sigma - 1 > \theta$.

To build intuition for these parameter conditions, consider a symmetric version of our economy where $\xi_{i\ell n} = \xi$. In this economy, the marginal value of location ℓ is given by:

$$D_\ell \pi_i (\mathcal{L}, z) = \mathcal{G} z^{\sigma-1} \left(|\mathcal{L} \cup \{\ell\}|^{\frac{\sigma-1}{\theta}} - |\mathcal{L} \setminus \{\ell\}|^{\frac{\sigma-1}{\theta}} \right) - w_\ell f_{i\ell}\,, \tag{8}$$

where $\mathcal{G}$ is a composite general equilibrium constant and, since all locations are symmetric, only the *number* of locations matters, not their precise identity. The cost and demand elasticities interact to determine the strength and direction of the complementarities among production locations. Equation (8) shows that if $\sigma - 1 > \theta$, the marginal value of any given production location ℓ is increasing in the number of other locations, so that SCD-C from below holds, indicating positive complementarities. If instead $\sigma - 1 < \theta$, complementarities are negative and SCD-C from above holds. In the special case where $\sigma - 1 = \theta$, the marginal value of ℓ is independent of the firm's decision set $\mathcal{L}$.[11] Finally, SCD-T requires that a location's marginal value is higher at more productive firms, which is the case as long as $\sigma > 1$.

Equation (8) highlights that firms in our model face a CDCP because of the combination

[11] In the Online Appendix, we also show how to verify SCD-C and SCD-T for more general marginal cost functions $c_{in} (\mathcal{L}, z)$ of which the cost function in this section is a special case. Our generalized framework nests production structures as in Ramondo (2014), Arkolakis, Ramondo, et al. (2018), Lind and Ramondo (2023), and Xiang (2022), that is, production with correlated idiosyncratic shocks across locations, or with multiple different technological methods of production. In general, regardless of the cost function, the strength of the scale complementarities depends on how changes in marginal costs translate into differences in variable profits, which is determined by the elasticity of demand and the pass-through elasticity of costs to price. In the present model, markups are constant so that the pass-through elasticity is always 1; consequently, only the elasticity of demand matters.

of two ingredients: complementarities among locations and the fixed costs of setting up production locations. Without complementarities, that is when $\theta = \sigma - 1$, the marginal value of any production location is independent of all others and hence the firm faces $|L|$ independent decisions. Without fixed costs, that is, when $f_{i\ell} = 0 \, \forall i, \ell$, the marginal value of all locations is always positive so that all firms establish production locations in all countries, regardless of complementarities.

4. Quantification

In this section, we provide an overview of our model calibration, which follows standard techniques and borrows central parameters from the literature. For use in our quantitative exercises in Section 5, we calibrate the model twice, once with negative and once with positive complementarities. We relegate most details of the quantification, such as of the treatment of the data and the model's fit, to Appendix C.

4.1. Data

We obtain information on manufacturing trade and multinational production for 32 countries from Alviarez (2019). For each host country, the data set contains the number of manufacturing enterprises and their total manufacturing sales by origin country. For example, the data contain the number of German manufacturing enterprises in France and their total sales. The dataset also contains bilateral trade flows for all country pairs. For all variables, the data reflect average values over the period from 2003 to 2012.[12]

While the dataset contains foreign affiliate counts for each nation, it lacks each country's total enterprise count and information on firm entry and survival. To supplement the dataset, we obtain counts of total enterprises for each country from the OECD Structural Statistics of Industry and Services dataset, and data on the 1-year survival rates of enterprises in each country from the OECD Structural and Demographic Business Statistics.

To construct standard bilateral gravity controls, we use the CEPII database (see Conte, Cotterlaz, Mayer, et al. 2023), which provides measures of geographic distance, a shared border indicator, a colonial ties dummy, and a common language dummy. In addition, we use the TRAINS dataset to construct bilateral manufacturing tariff measures, following standard practices in the literature. Data on real GDP per capita and total employment are drawn from

[12] To construct the dataset, Alviarez (2019) combined data from the OECD, the Eurostat Foreign Affiliate Statistics database, the Bureau of Economic Analysis, and Bureau van Dijk's Orbis dataset. The dataset contains information on the following countries: Australia, Austria, Belgium, Bulgaria, Canada, the Czech Republic, Denmark, Estonia, Finland, France, Germany, Greece, Hungary, Italy, Japan, Latvia, Lithuania, Mexico, the Netherlands, New Zealand, Norway, Poland, Portugal, Romania, the Russian Federation, Slovakia, Spain, Sweden, Turkey, Ukraine, the United Kingdom, and the United States.

Parameter	Description	Value/Target Moment
Externally Calibrated		
σ	Demand elasticity	Set to 4 (Arkolakis, Ramondo, et al. 2018)
$\frac{\sigma-1}{\theta}$	Location complementarity	Set to $\frac{2}{3}$ (negative complementarities) and $\frac{3}{2}$ (positive complementarities)
Internally Calibrated		
ζ	Firm Pareto shape	Sales distribution right-tail (Arkolakis 2010)
$\underline{z}_i$	Firm Pareto minimum	Total foreign MP (Alviarez 2019)
T_ℓ	Location productivity	GDP per capita (PWT)
f_i	Fixed cost (origin component)	1-year firm survival rate (OECD)
f_i^e	Entry cost	Total enterprise counts (OECD)
H_ℓ	Labor supply	Employment (PWT)
$\tau_{\ell n}, \gamma_{i\ell}, \nu_{i\ell}$	Bilateral trade, MP, fixed costs	Gravity coefficients on trade flows, MP flows, affiliate counts (Appendix C.2)

Table 1: Parameters and Target Moments

This table summarizes the calibration strategy of the model. Two parameters, σ and θ, are externally calibrated; the remaining are calibrated simultaneously in a single iterative routine that targets the moments described in the last column. For each empirical moment, we list the data source. PWT refers to the Penn World Tables and OECD to either the OECD Structural Statistics of Industry and Services dataset or for the 1-year survival rates of enterprises in the OECD Structural and Demographic Business Statistics.

the Penn World Tables (Feenstra, Inklaar, and Timmer 2015).

4.2. Calibration Strategy

Our calibration strategy uses indirect inference to identify the model's internally calibrated elasticities. For a given set of elasticity values, we choose productivity parameters to match country-specific aggregates in general equilibrium. Once these targets are met, we use the model's equilibrium allocations to estimate a set of aggregate gravity regressions and update the elasticity guesses to match the corresponding coefficients in the data. Since matching country-level aggregates in general equilibrium is itself an iterative process that requires repeatedly solving for the policy function, our method is essential to the feasibility of this approach. Table 1 summarizes the moments targeted by each parameter. Further details are in the Online Appendix.

Demand Elasticity and Location Substitution Elasticity: σ and θ Together, the elasticity of substitution σ across consumption goods and the elasticity of substitution θ across production locations in the firm's cost function determine the type of SCD-C condition satisfied by the firm location problem. In our baseline calibration, we follow Arkolakis, Ramondo, et al. (2018)

and set $\sigma = 4$.[13]

We calibrate the model twice, with two different values for θ, to conduct quantitative exercises with both negative and positive complementarities. For the negative complementarity calibration, we follow Arkolakis, Ramondo, et al. (2018) and set $\theta = 4.5$. Combined with the value for σ, the degree of complementarity is $\frac{\sigma-1}{\theta} = \frac{2}{3}$. For symmetry, we set θ in the positive complementarity calibration so that $\frac{\sigma-1}{\theta} = \frac{3}{2}$.

Country Productivity Parameters and Fixed Costs We assume that the firm productivity distribution $G_i(z)$ is Pareto with shape parameter ζ and minimum $\underline{z}_i$ which is specific to each headquarter country i. The model generates a firm sales distribution with a Pareto tail of $\zeta/(\sigma-1)$, so we set $\zeta = 1.65 \times (\sigma-1)$ to match the tail estimate presented in Arkolakis (2010).

The model features two vectors of country-specific productivity terms: the minimum of the firm Pareto distribution, $\underline{z}_i$, which acts as a location-of-headquarter productivity shifter; and the location-of-production productivity shifter, T_ℓ. We choose the first to exactly match the share of global foreign production attributable to firms headquartered in i, and the second to exactly match the observed GDP per capita for each country.

For the calibration, we decompose the fixed cost of setting up a production location in ℓ for firms headquartered in location i into an origin-specific shifter and a bilateral term, $f_{i\ell} \equiv f_i \nu_{i\ell}$. For a given entry cost f_i^e, we choose the base component of fixed costs f_i to match the share of "surviving" firms whose productivity draw exceeds the cutoff level defined by the zero profit condition in equation (3). In a second step, we recover the entry cost f_i^e by inverting the free entry condition in equation (4) so that the model matches the implied number of entering firms observed in the data, which we calculate as the number of enterprises divided by the survival rate. We set total labor supply in each location, H_i, to match total employment in each country.

Trade Costs, MP Costs, Fixed Costs In the theory, three bilateral cost matrices shape the patterns of trade flows, foreign affiliate sales, and foreign affiliate counts across country pairs: the matrix of trade costs $\{\tau_{\ell n}\}_{\ell n}$, the matrix of MP costs $\{\gamma_{i\ell}\}_{i\ell}$, and the matrix of the bilateral components of fixed costs $\{\nu_{i\ell}\}_{i\ell}$. To calibrate these parameters, we use data on trade flows, MP flows, and bilateral affiliate counts as indicated in Table 1.

[13] Our choice of σ falls within the range of estimates of Broda and Weinstein (2006). Arkolakis, Ramondo, et al. (2018) show that it is also consistent with markup estimates from the manufacturing sector. Moreover, our value is similar to the estimate of $\sigma = 3.89$ from Head and Mayer (2019). While Head and Mayer (2019) focuses on multinational production in the car industry, their estimation strategy is consistent with our theoretical setup and also generates markups in line with the microeconomic estimates from the car industry in Goldberg (1995) and Berry, Levinsohn, and Pakes (1995). Finally, our choices of θ generates a trade elasticity of 2.7. Head and Mayer (2014) report a median estimate of the trade elasticity of 3.19 and a mean of 4.51 based on a meta-analysis of 32 papers.

We follow Tintelnot (2017) and parameterize the three matrices $\{\tau_{\ell n}, \gamma_{i\ell}, \nu_{i\ell}\}$ as constant elasticity functions of the standard bilateral gravity variables: geographic distance, a shared colonial past indicator, a shared language indicator, and a common border indicator. We specify a separate elasticity for every gravity variable in each of the three matrices, leading to a total of twelve elasticities to estimate. We present the expressions for each element of the bilateral cost matrices in Appendix C.2.[14] We then choose the elasticities of the various gravity variables in the bilateral cost matrices using an indirect inference procedure. We specify three gravity equations, for the trade flows, MP flows, and bilateral affiliate counts both in model and data and choose the elasticities of the gravity variables in the model to match the regression coefficients found in the data.

5. Quantitative Exercises and Counterfactuals

In this section, we use our quantified model to conduct a number of numerical exercises illustrating the performance of our solution method. We also use our model to measure the welfare gains from multinational production, in particular analyzing how they differ under negative or positive complementarities and in settings with or without fixed costs.

5.1. Computational Performance

We conduct three different numerical experiments, demonstrating advantages of our solution method in terms of computational speed, precision, and breadth of applicability. For experiments that vary the number of countries, we sample from the the cost and productivity parameters of our calibrated model to generate synthetic countries.[15]

Speed We compare the speed of our policy function method ("Policy") against two alternative discretization approaches on a grid of $2^{14} \approx 16000$ grid points: the "naive" approach which uses brute force to solve the CDCP at each grid point by evaluating profits for all possible location combinations and the "squeezing" approach which applies the squeezing procedure at each grid point until convergence, then brute force to the reduced domain. Unlike our "policy" method, these two grid-based approaches generate discretization error in the computed policy functions.

Table 2 reports the time (in seconds) required to compute the policy function across varying numbers of synthetic countries. We compute $\mathcal{L}_i^{\star}(z)$ for firms from each origin i and report the average time across origins. The naive method requires more than an hour even with

[14] For any bilateral pair without MP activity in the data, we set $\gamma_{i\ell} = \infty$.

[15] A synthetic country is described by fundamentals $\{T_\ell, f_i\}$, aggregates $\{w_\ell, P_\ell\}$, and bilateral costs $\{\tau_{\ell n}, \gamma_{i\ell}, \nu_{i\ell}\}$. For each of these objects, we non-parametrically fit a distribution to the estimated values from which we then sample to generate synthetic countries.

Countries	Negative Complementarities			Positive Complementarities		
	Naive (1)	Squeezing (2)	Policy (3)	Naive (1)	Squeezing (2)	Policy (3)
8	8	0.423	0.019	7	0.480	0.034
16	5454	2.261	0.039	4356	2.364	0.087
32	–	11.093	0.107	–	13.217	0.186
64	–	66.001	1.323	–	94.459	1.293
128	–	456	14.061	–	795	14.702
256	–	3239	331	–	6479	374
Grid points	2^{14}	2^{14}	–	2^{14}	2^{14}	–

TABLE 2: RUNTIMES FOR DIFFERENT SOLUTION METHODS

This table illustrates the computational time in seconds for computing the firm policy function by discretizing firm productivity, then solving the firm's CDCP at each grid point by (1) evaluating the profit function for all possible production location combinations ("Naive"); or (2) applying squeezing, then brute force ("Squeezing"). We contrast these methods with our policy function method ("Policy") which does not require discretization. We repeat the exercise in both the calibration with negative and positive complementarities. Synthetic countries are generated by sampling from the distribution of parameter estimates obtained from the calibrated model. We then time how long it takes to solve for the policy functions $\mathcal{L}_i^\star(\cdot)$ for all origins, and report the average time taken across origins. Trials were computed on an Apple M1 (2020) CPU.

just 16 countries, while the policy method solves the same problem in less than a tenth of a second. Even for 128 countries, the policy method recovers the average policy function in under 15 seconds, while it takes about 6 minutes to compute the average policy function with 256 countries.

In the case of positive complementarities, the squeezing method corresponds to the incumbent approach pioneered by Antras, Fort, and Tintelnot (2017), and our policy function method is roughly an order of magnitude faster across all country counts. For negative complementarities, the naive brute-force approach represents the benchmark, and our method improves on it by four orders of magnitude. Approximately three-quarters of the speed gains come from extending the squeezing method of Jia (2008) to the negative complementarity case, and the remaining quarter from introducing our policy function method, which avoids solving the CDCP at every grid point.

Precision Table 2 understates the performance gap between the policy method and the alternative approaches, as it does not account for the discretization error inherent in the latter. Discretization error arises because outcomes must be interpolated between grid points, so that selecting the number of grid points involves a trade-off between computational time and discretization error.

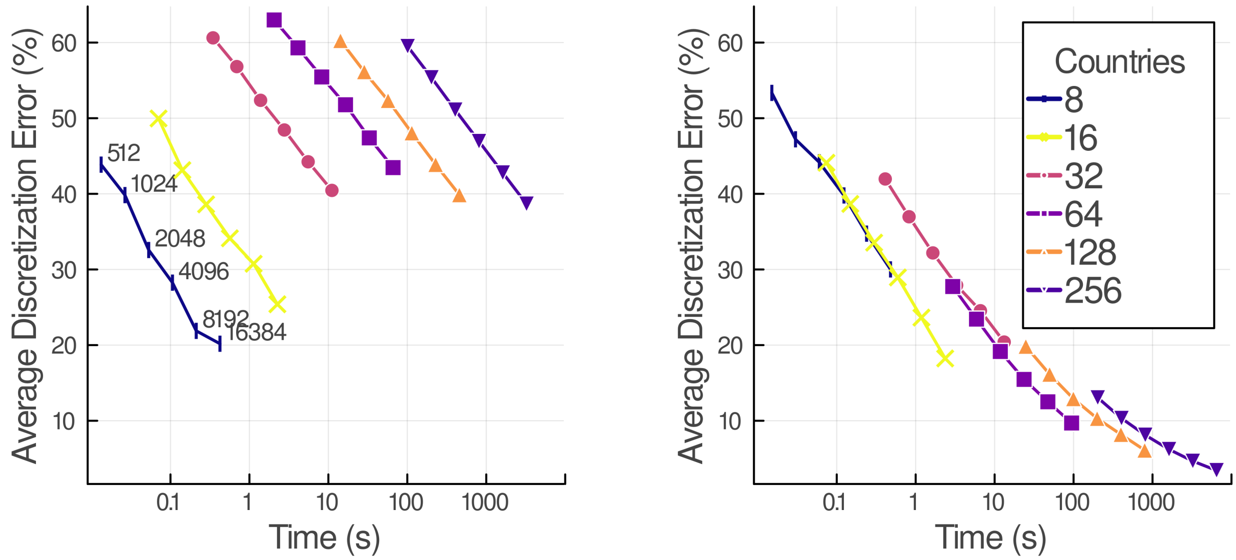
Our policy function method solves for the *exact* policy function, allowing us to quantify the discretization error of other methods. To do so, we first compute the total value of trade flows from origin i to destination n via production in country ℓ, denoted $X_{i\ell n}$, by aggregating over the exact policy function. Then, we compute the corresponding approximation $\hat{X}_{i\ell n}$ by aggregating over the discretized policy function of the "Squeezing" method, interpolating between grid points. We define the discretization error as the average percentage deviation of flows across all country triplets, that is, $N^{-3}\sum_{i,\ell,n}\left|\hat{X}_{i\ell n}/X_{i\ell n}-1\right| \times 100\%$.

Figure 4 graphs the discretization error against computational time for varying grid densities and both types of complementarities. Each line shows the precision-time frontier for a different number of countries. With 512 grid points and 8 countries, computation is fast, but the average discretization error exceeds 40% regardless of the type of complementarity. Even with over 16000 grid points, the error remains above 20% in both cases. Doubling the number of grid points reduces the error by 5 to 10 percentage points but doubles the computation time. As the number of countries increases, the frontier shifts rightward at a log-constant rate, reflecting the polynomial complexity of the generalized squeezing procedure. With negative complementarities, average discretization error increases in the number of countries. By contrast, it decreases with positive complementarities, since the nesting structure of the policy function mitigates discretization error.

Wide Applicability In a final exercise, we examine how computational time depends on the strength and direction of complementarities, summarized by the ratio $(\sigma-1)/\theta$. We recalibrate the 32-country model (following Section 4) for values of θ ranging from 0.8 to 20, yielding $(\sigma-1)/\theta$ values between 0.15 and 3.9. This range extends substantially beyond the strength of complementarities suggested by estimates of θ and σ in the literature and notably also includes cases where $0<\theta<1$, so that production locations are complements in the firm's cost function.

The left panel of Figure 5 shows the average time required to solve for the policy function across origin countries, using our policy function method for every level of complementarity. The figure also shows the fraction of the total computational time spent using the squeezing method. Across the range of complementarities, our policy function method takes under 0.26 seconds. Computation is fastest near the vertical line, which marks the special case of no complementarity, where production locations are effectively independent. As the degree of complementarity increases in either direction, computation time increases but never more than doubles.

Our theoretical results help explain the patterns in the left panel. Theorem 2 establishes that the generalized squeezing method never takes more than $|L|$ applications, consistent with little



(A) Negative Complementarities

(B) Positive Complementarities

FIGURE 4: THE PRECISION-TIME FRONTIER WITH DISCRETIZATION METHODS

This figure illustrates the discretization error in computing the policy function with the "Squeezing" approach, for different numbers of countries. To measure discretization error, we compute the percentage deviation for each trilateral flow $X_{i\ell n}$, when using the discretized policy function compared to the true policy function. In particular, we compute the average discretization error as $N^{-3}\sum_{i,\ell,n}\left|\hat{X}_{i\ell n}/X_{i\ell n}-1\right| \times 100\%$, where $X_{i\ell n}$ are the sales of location ℓ production sites, whose headquarters are in location i, to destination market n computed using the "Policy" method (that is, the exactly solved model) and $\hat{X}_{i\ell n}$ is the same object in the model computed using the "Squeezing" method. Trials were computed on an Apple M1 (2020) CPU.

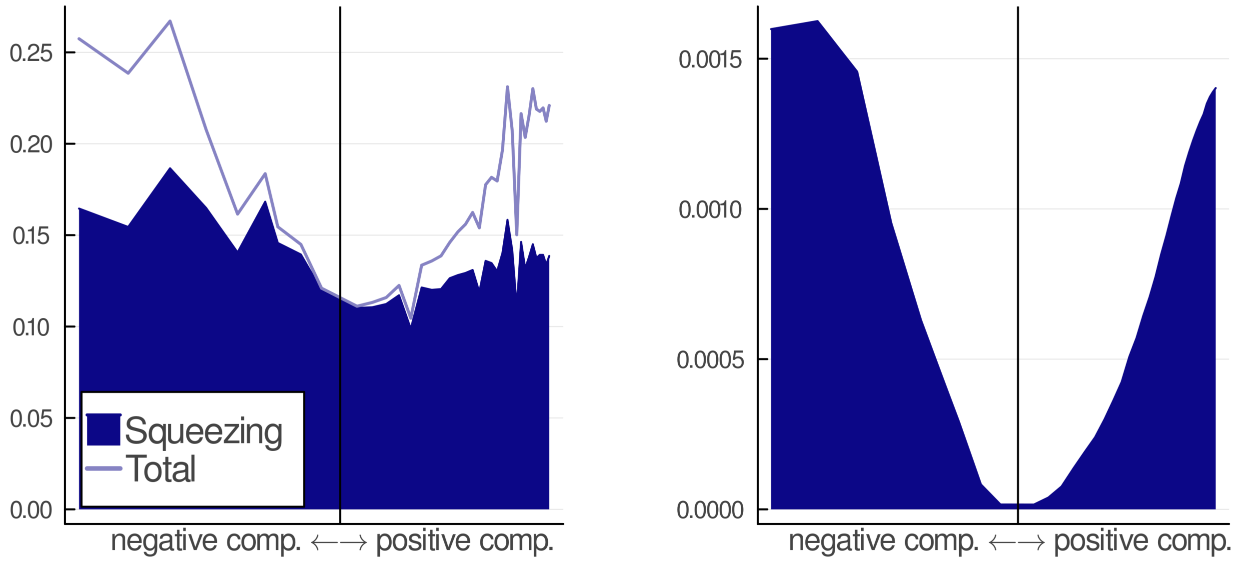
variation in squeezing time across the full range of complementarities.

The right panel shows a proxy measure for the effectiveness of the squeezing procedure: the number of locations which separate the upper and lower bounding set averaged across all firms and countries after convergence of the generalized squeezing procedure. In general, there are very few locations separating the bounding sets after convergence. At the same time, the average number of leftover locations increases with stronger complementarities, consistent with the increasing gap between squeezing and total time in the left panel as complementarities grow stronger.

5.2. CDCPs and the Welfare Gains from Multinational Production

Complementarities among production locations, together with the fixed costs of setting them up, give rise to the CDCP in our model. An alternative modeling approach is to abstract from either complementarities or fixed costs to avoid solving CDCPs. In this section, we examine how the welfare gains from multinational production depend on the presence of both forces.

To quantify the role of these assumption for the welfare gains from multinational production,



(A) Compute Time (s)

(B) Average Value of $\left|\overline{\mathcal{L}}^{(K')}(\cdot)\setminus\underline{\mathcal{L}}^{(K')}(\cdot)\right|$

FIGURE 5: COMPLEMENTARITIES AND COMPUTATIONAL PERFORMANCE

This figure shows the performance of the policy function method with different degrees of complementarity. The strength of complementarity is measured by the ratio $\frac{\sigma-1}{\theta}/\left(1+\frac{\sigma-1}{\theta}\right)$, and varied by changing θ while holding $\sigma=4$ fixed (its value in the baseline calibration). The vertical line indicates the case with no complementarities, with $\frac{\sigma-1}{\theta}=1$. The panel on the left shows the total policy function method computation time in seconds as well as the time spent on generalized squeezing in particular. The right panel shows the number of locations which separate the upper and lower bounding set, averaged across all firms and countries, after convergence of the generalized squeezing procedure. Trials were computed on an Apple M1 (2020) CPU.

we extend the welfare formula in Arkolakis, Costinot, and Rodríguez-Clare (2012) to the context of our model with both complementarities and fixed costs.[16] The following equation describes the welfare impact in country i of imposing trade or MP autarky by setting the respective bilateral costs to infinity:

$$\ln\frac{\hat{w}_i}{\hat{P}_i}=\underbrace{\ln\hat{\pi}_{iii}^{-\frac{1}{\sigma-1}}}_{\text{openness}}+\underbrace{\ln\hat{M}_i^{\frac{1}{\sigma-1}}+\ln\hat{\bar{z}}_i^{-\frac{\zeta}{\sigma-1}}}_{\text{varieties}}+\underbrace{\ln\hat{\bar{z}}_i+\ln\left[\sum_{\mathcal{Z}_i^t\in\mathcal{T}_i}\lambda_{iii}^t\left(s_{iii}^t\right)^{\frac{\sigma-1}{\theta}-1}\right]^{\frac{1}{\sigma-1}}}_{\text{average productivity}} \tag{9}$$

where $\hat{x}=x'/x$ and x denotes the value of a variable in our baseline calibration and x' its value under autarky. We denote by π_{iii} the fraction of all final spending in i on goods produced

[16]We derive this welfare formula in Appendix A under the assumption that, in the initial equilibrium, all active firms include the headquarter location in their optimal set of production locations. This assumption holds in all numerical exercises.

in i by firms headquartered in i. This own share is an inverse measure of openness of i to *both* trade and multinational activity. M_i is the mass of entrants in location i and $\tilde{z}_i$ is the survival productivity cutoff in location i for firms headquartered in i. The policy function $\mathcal{L}_i^\star(\cdot)$ induces a partitioning $\mathcal{T}_i = \{\mathcal{Z}_i^t\}_t$, so that $\mathcal{L}_i^\star(z) = \mathcal{L}_i^t$ for all $z \in \mathcal{Z}_i^t$. Out of all sales in i by firms from i in interval t, the term s_{iii}^t denotes the share that is produced in i so that $s_{i\ell i}^t$ sums to 1 across production locations ℓ, within each interval t. Out of total sales in i by firms from i produced in i, λ_{iii}^t denotes the share accounted for by firms in interval t, so that λ_{iii}^t sums to 1 across all intervals t.

Equation (9) shows that the welfare effects of moving to trade or MP autarky work through three channels: first, a standard *openness channel* that captures a reduction in real consumption; second, a *varieties channel* that adjusts for changes in the number of domestic varieties; and third, an *average productivity channel* that adjusts for changes in the average productivity with which domestic goods are produced.

Since trade and MP autarky shift inward the effective production possibility frontier of all countries, the openness channel is typically negative. The signs of the variety and productivity channels, however, tend to depend on the degree to which countries are headquarter locations for multinationals compared to host countries for the foreign affiliates of multinationals headquartered abroad.

The variety effect reflects that both trade and MP autarky shrink the profits of previously large firms engaged in these foreign activities. The resulting reduction in local wages allows the selection cutoff to fall and more local firms to survive, creating new varieties that benefit welfare. Relative to the no-complementarity case, firms engaged in MP are on average larger and more profitable in the positive complementarity case than in the negative complementarity case, since complementarities directly shape both the marginal cost advantage attainable through MP and the market size advantage attainable through trade. As a result, in the case of positive complementarities, the variety effect is more positive compared to the case of negative complementarities.

The average productivity channel captures both extensive and intensive margin effects of the move to autarky. On the extensive margin, average firm productivity declines because the entry threshold shifts downward ($\hat{\tilde{z}}_i < 1$), allowing lower-productivity firms to survive. On the intensive margin, captured by the last term in the expression, the sales-weighted average productivity of operating firms changes as relative firm sizes adjust. When locations are complements ($\sigma - 1 > \theta$), the most productive firms in each origin country i shrink the most, as they lose the scale economies that previously amplified their productivity advantage and supported their large domestic market share. As a result, reallocation among incumbents in the domestic market implies that the sales-weighted average productivity declines and

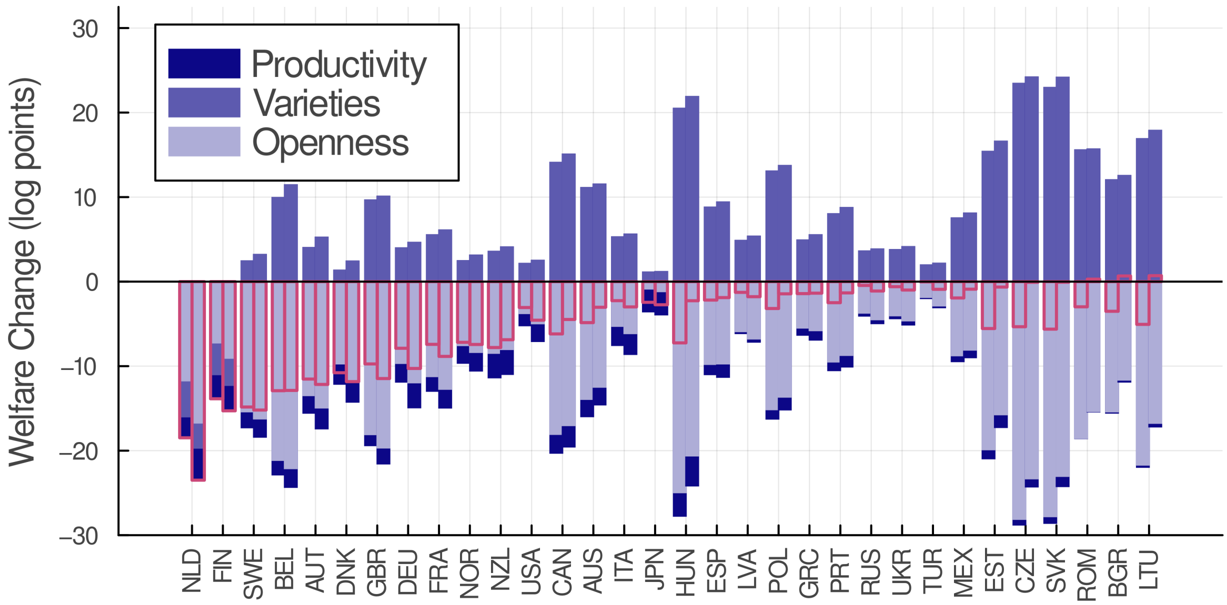


FIGURE 6: WELFARE CONSEQUENCES OF IMPOSING MP AUTARKY

This figure shows the log point welfare change ($100 \times \ln\left(\hat{w}_i/\hat{P}_i\right)$) from moving from the calibrated economy to MP autarky as pink outlines. In addition, the figure decomposes the welfare changes into the contributions from changes in openness, changes in the number of available varieties, and changes in average productivity from equation (9). The left bars are for the calibration with negative complementarities, and the right bars with positive complementarities. The countries are ordered by the size of the total welfare effect in the positive complementarity calibration.

the intensive margin effect is always negative. By contrast, when locations are substitutes ($\sigma - 1 < \theta$), the most productive firms expand relative to other firms in the domestic market, as they substitute foreign production with domestic production. Thus, the sales-weighted average productivity increases and the intensive margin effect is always positive.

We now use the two calibrations of our model to study the welfare consequences of moving to MP autarky with negative versus positive complementarities. We focus on MP autarky because it highlights the role of multinational location choices with complementarities and fixed costs, that our new solution methods make tractable. Figure 6 shows the total welfare change of moving to MP autarky in our calibrated models in pink while the blue bars decompose the total effect into the three channels of equation (9). Countries are ordered by the size of the total welfare effect in the calibration with positive complementarities. For each country, the left bars show results from the calibration with negative complementarities while the right bars show results from the calibration with positive complementarities.

Figure 6 shows that the vast majority of countries suffers negative welfare consequences from removing multinational production. Effects range from reductions in welfare of 20 log points for productive and open economies like the Netherlands, to negligible changes for some less productive and less open economies like Turkey. In general, the effect is less negative with negative complementarities compared to positive complementarities. For some less productive countries, such as Bulgaria, the opposite is true: with positive complementarities, there are welfare *gains* from moving to MP autarky while there are welfare losses under negative complementarities.

The countries on the left in Figure 6 are the main benefactors of multinational production: these are small, productive, and open economies in which many multinationals are headquartered. In these countries, like all countries, the openness channel decreases real consumption in the move to MP autarky. Moreover, their most productive firms lose the marginal cost advantages from multinational production, lowering average productivity substantially, especially with positive complementarities. In fact, some of these countries actually lose varieties, indicating that the low marginal costs attainable through foreign production were central in enabling firms with low productivity draws to make a positive profit; once these marginal cost advantages are no longer available, the number of surviving profitable domestic varieties shrinks.

By contrast, the countries on the right are small, low-wage locations with few domestic firms productive enough to engage in multinational production. Instead, they serve primarily as the production sites of productive firms headquartered elsewhere. The openness channel implies real consumption declines as the world production possibility shifts in and prices increase. However, these losses are offset by a positive variety effect: the labor released by departing multinationals crowds in the creation of new domestic varieties. The variety effect is stronger with positive complementarities, since multinational firms lose more of their marginal cost advantage when constrained to domestic-only production, shrinking and releasing more labor in the process. The average productivity effect is small for these countries, since the fact that so few domestic firms engage in multinational production implies that the term in square brackets is close to 1 as $s^t_{iii} \approx 1$ for all t. In turn, because relative firm sizes change little, the productivity cutoff remains (almost) unchanged and released labor predominantly increases entry.[17]

Overall, our analysis shows that calibrating the model with different types of complementarity *to the same data* yields substantially different welfare effects of MP autarky, with sign reversals in some cases.

Fixed costs are also essential: without fixed costs of establishing production locations, $\hat{\bar{z}}_i = 1$, and all firms produce in all countries. In this case, since all firms are affected symmetrically,

[17]If no domestic firm engages in multinational production, the distribution of domestic firms can be described by a representative firm as in the closed economy of Melitz (2003). In this case, as a consequence of CES demand together with Pareto productivity, changes in MP costs cause no reallocations among firms, so that the productivity of the representative firm does not change.

the term in square brackets collapses to a common share s_{iii} that captures labor reallocation between variable production and firm entry. In calibrations without fixed costs, welfare losses from MP autarky are substantially larger for nearly all countries, with either level of complementarity (see Figure 12 in the Online Appendix). Intuitively, with fixed costs, the retreat of multinational activity frees up labor that can be reallocated to the creation of new domestic varieties, partially offsetting the welfare loss. Thus, models that abstract from fixed costs tend to overstate the welfare losses from MP autarky.

In sum, our results suggest that modeling complementarities in location decisions—together with fixed costs—is crucial for quantitatively evaluating counterfactuals involving multinational production.

6. Conclusion

We introduce a method to solve combinatorial discrete choice problems with either negative or positive complementarities, and across heterogeneous agents. We apply it to a quantitative model of multinational production in which heterogeneous firms choose sets of foreign production locations. Our approach allows us to solve and calibrate the model in general equilibrium with many locations. The calibrated model shows that complementarities and fixed costs—central to the firms' combinatorial problem—are also critical for evaluating the gains from multinational production. Gains are larger with positive than with negative complementarities, and smaller when fixed costs are included. Models that abstract from these features may misstate the effects of disruptions that affect the cost of multinational production.

Beyond multinational production, our methods apply to a broad range of problems with interdependent discrete choices, for example in the context of discrete infrastructure investments, supply chain formation, or the location decisions of multi-establishment firms. By making such problems computationally tractable, our approach expands the frontier of applied general equilibrium analysis and enables counterfactuals that were previously deemed infeasible.

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A. Counterfactual welfare

The trilateral sales in the model are given by

$$X_{i\ell n} = \tilde{\zeta} M_i X_n \left(\frac{P_n}{\mu}\right)^{\sigma-1} \sum_{\mathcal{Z}_i^t \in \mathcal{T}_i} \mathbb{1}_{i\ell}^t \xi_{i\ell n}^{-\theta} \left(\Theta_{in}^t\right)^{\frac{\sigma-1}{\theta}-1} \underline{z}_i^{\zeta} \left[\left(z_i^{t+1}\right)^{\sigma-1-\zeta} - \left(z_i^t\right)^{\sigma-1-\zeta}\right]$$

where $\tilde{\zeta}$ is a constant of integration and, for brevity, we denote $\Theta_{in}^t \equiv \sum_{\ell \in \mathcal{L}_i^t} \xi_{i\ell n}^{-\theta}$. Then, the counterfactual change is

$$\hat{X}_{i\ell n} = \hat{M}_i \hat{w}_n \hat{P}_n^{\sigma-1} \frac{\sum_{\mathcal{Z}_i^{t\prime} \in \mathcal{T}_i^{\prime}} \mathbb{1}_{i\ell}^{t\prime} \xi_{i\ell n}^{\prime -\theta} \left(\Theta_{in}^{t\prime}\right)^{\frac{\sigma-1}{\theta}-1} \left[\left(z_i^{t+1\prime}\right)^{\sigma-1-\zeta} - \left(z_i^{t\prime}\right)^{\sigma-1-\zeta}\right]}{\sum_{\mathcal{Z}_i^t \in \mathcal{T}_i} \mathbb{1}_{i\ell}^t \xi_{i\ell n}^{-\theta} \left(\Theta_{in}^t\right)^{\frac{\sigma-1}{\theta}-1} \left[\left(z_i^{t+1}\right)^{\sigma-1-\zeta} - \left(z_i^t\right)^{\sigma-1-\zeta}\right]}.$$

Computing the counterfactual change in X_{iii} derives the welfare formula. To do so, we adopt the following assumption for the remainder of the section.

Assumption. *Model fundamentals are such that, in the baseline equilibrium, it is the case that* $\mathbb{1}_{ii}(z) = 1$ *for all* $z \geq \tilde{z}_i$, *that is, all active firms establish production in their country of origin* i.

This assumption guarantees that no firms produce in foreign locations but not domestically. With either MP or trade autarky, the change simplifies to

$$\begin{aligned}
\hat{X}_{iii} &= \hat{M}_i \hat{w}_i \hat{P}_i^{\sigma-1} \hat{\xi}_{iii}^{-\theta} \frac{\left(\xi_{iii}^{\prime -\theta}\right)^{\frac{\sigma-1}{\theta}-1} \left(\tilde{z}_i^{\prime}\right)^{\sigma-1-\zeta}}{\sum_{\mathcal{Z}_i^t \in \mathcal{T}_i} \mathbb{1}_{ii}^t \left(\Theta_{ii}^t\right)^{\frac{\sigma-1}{\theta}-1} \left[\left(z_i^t\right)^{\sigma-1-\zeta} - \left(z_i^{t+1}\right)^{\sigma-1-\zeta}\right]} \\
&= \hat{M}_i \hat{w}_i \hat{P}_i^{\sigma-1} \hat{\xi}_{iii}^{1-\sigma} \hat{\tilde{z}}_i^{\sigma-1-\zeta} \sum_{\mathcal{Z}_i^t \in \mathcal{T}_i} \lambda_{iii}^k \left(s_{iii}^t\right)^{\frac{\sigma-1}{\theta}-1},
\end{aligned}$$

where we use the fact that $\hat{\xi}_{i\ell n}^{-\theta} = 0$ for all $i \neq \ell$ and define

$$s_{i\ell i}^t \equiv \frac{\xi_{i\ell i}^{-\theta}}{\sum_{\mathcal{L}_i^t} \xi_{i\ell i}^{-\theta}} \quad , \quad \lambda_{iii}^k \equiv \frac{\mathbb{1}_{ii}^t s_{iii}^t \left(\Theta_{ii}^t\right)^{\frac{\sigma-1}{\theta}} \left[\left(z_i^t\right)^{\sigma-1-\zeta} - \left(z_i^{t+1}\right)^{\sigma-1-\zeta}\right]}{\sum_{\mathcal{Z}_i^t \in \mathcal{T}_i} \mathbb{1}_{ii}^t s_{iii}^t \left(\Theta_{ii}^t\right)^{\frac{\sigma-1}{\theta}} \left[\left(z_i^t\right)^{\sigma-1-\zeta} - \left(z_i^{t+1}\right)^{\sigma-1-\zeta}\right]}.$$

The first is the share of production, sold in i and produced by interval $\mathcal{Z}_i^t$, that takes place in location ℓ compared to other locations. The second is the share of production X_{iii} that is accounted for by interval $\mathcal{Z}_i^t$. Using the fact that $\hat{\xi}_{iii}^{1-\sigma} = \hat{w}_i^{1-\sigma}$, then rearranging, we arrive at

$$\frac{\hat{w}_i}{\hat{P}_i} = \hat{\pi}_{iii}^{\frac{1}{1-\sigma}} \hat{M}_i^{\frac{1}{\sigma-1}} \hat{\bar{z}}_i^{1-\frac{\zeta}{\sigma-1}} \left[\sum_{\mathcal{Z}_i^t \in \mathcal{T}_i} \lambda_{iii}^t \left(s_{iii}^t\right)^{\frac{\sigma-1}{\theta}-1} \right]^{\frac{1}{\sigma-1}} .$$

B. Input Microfoundation for CES Marginal Cost

In this section, we lay out a microfounded motivation for the CES marginal cost function employed in the main body of the paper that follows Antras, Fort, and Tintelnot (2017).

Firms produce their final good by combining a continuum of firm-specific intermediate inputs, indexed by v, with a constant elasticity of substitution η. Each of the firm's production locations can produce the entire continuum of intermediate inputs.

For a firm headquartered in location i, the marginal cost of producing an input variety v at a production site in location ℓ is given by $\gamma_{i\ell} w_\ell / \varphi_\ell(v)$, where $\varphi_\ell(v)$ is a location-input-specific productivity shock and $\gamma_{i\ell}$ is a bilateral cost of multinational production. For each destination n and intermediate input v, the firm chooses from among its set of production sites, $\mathcal{L}$, the location $\ell_{in}^\star(\boldsymbol{\varphi}(v))$ that offers the lowest destination-specific marginal cost:

$$\ell_{in}^\star(\boldsymbol{\varphi}(v)) = \arg\min_{\ell \in \mathcal{L}} \gamma_{i\ell} \frac{w_\ell}{\varphi_\ell(v)} \tau_{\ell n},$$

where the term $\tau_{\ell n}$ denotes a bilateral iceberg trade costs and the vector $\boldsymbol{\varphi}(v) = \{\varphi_\ell(v)\}_\ell$ collects a firm's productivity of producing input v in every location ℓ.

Suppose the firm draws each of the productivity terms $\varphi_\ell(v)$ independently from a Fréchet distribution with shape θ and scale $T_\ell z$, *after* making its production location decision $\mathcal{L}$. Then, the Fréchet distribution on idiosyncratic location draws implies the CES cost function used in the main body of the paper, up to a constant of integration. The substitutability among plants derives from the fact that plants cannibalize one another's sales as they compete to be the least cost supplier. The strength of this force depends on how much production locations differ in their productivity at producing any given variety as measured by the dispersion ($1/\theta$) of the location-input-specific productivity shocks. If comparative advantage differences among production locations are large ($1/\theta$ is large), the substitutability across locations is low and cannibalization is limited.

The properties of the Fréchet distribution imply that the expression in equation (1) is independent of the elasticity of aggregation across varieties η (see Eaton and Kortum 2002). A similarly tractable expression arises if, instead of the independent Fréchet distributions, each location-input-specific productivity shock is drawn from a multivariate correlated Fréchet or Pareto distribution with shape θ and correlation ρ as in Ramondo (2014) and Arkolakis, Ramondo, et al. (2018). Integrating across inputs delivers the CES cost function in (1), with θ replaced by $\frac{\theta}{1-\rho}$.

C. Data and Calibration

In this section, we discuss our data construction in more detail and provide additional information about the calibration.

C.1. Data

Trade, Foreign Affiliate Sales, and Foreign Affiliate Counts We use the data set compiled by Alviarez (2019) for our bilateral flow data. The data set combines information from four major databases: OECD International Direct Investment Statistics and the Statistics on Measuring Globalization; Eurostat Foreign Affiliate Statistics database; Bureau of Economic Analysis (BEA) public data; and Bureau van Dijk's Orbis dataset. All values in the data set are averages from 2003 to 2012; accordingly, in all other data sets we use in the calibration, we also take average values over the same period.

The data contains information for the following set of 32 countries for nine sectors. While we keep all the countries, we collapse all data across manufacturing industries to obtain manufacturing sector totals. For all combinations of these countries and sectors, the data contains the value of trade flows. We construct home absorption by summing a country's total export sales and subtracting them from the total sales of the sector in the country to obtain sales to the domestic market. Using these home sales, we can then construct home trade shares.

In addition, for each such origin-destination-sector triplet, the data contain the total sales of foreign affiliates, e.g. the total sales of Canadian companies located in Germany; note that there is no information on the destination countries of foreign affiliates in a given country, i.e. the data set does not report how much Canadian companies in Germany are selling to Greece. We construct sales of a country's domestic firms at home to destinations anywhere in the world by taking the total sales of all firms in the country and subtracting the total sales by foreign affiliates of other countries done in the country. Using these home sales, we can construct the complete matrix of (inward and outward) MP.

Lastly, the data contain information on the total foreign affiliates for each country-sector pair. The data do not contain information on the total domestic enterprises. We bring in data on the average number of total enterprises in each country between 2003 and 2012 (see below). We then subtract the total number of foreign affiliates operating in a country from the total number of enterprises and interpret the difference as the number of domestic enterprises or "headquarters" in our model. We can then once again compute the entire matrix of (inward and outward) MP enterprise shares for all country combinations.

CEPII Data We use the standard set of gravity variables from the CEPII database (see Conte, Cotterlaz, Mayer, et al. 2023) with minor modifications. As our distance measure, we use the simple distance between countries' most populated cities, measured in kilometers. To generate our colonial dummy variable, we combine the "colonial sibling" dummy, which indicates if two countries had a past common colonizer, and the "colonial dependence" dummy, which indicates if one country ever colonized the other from the CEPII data into one "colonial relationship" dummy. We also use the two dummy variables that indicate for every country pair whether it shares a border or an official language.

TRAINS Tariff Data We use information on tariffs from the TRAINS dataset for 2003–2012. This database reports the MFN (most favored nation) tariff rates for over 5,000 HS6 goods categories and country pair combinations. The data also contain information on preferential trade agreements and their postulated rates. We drop observations for which trade is subject to non-ad valorem (specific or nonlinear/compound) tariffs. For these tariffs, TRAINS reports ad-valorem equivalents. However, computation of these equivalents requires data on quantities, which are often noisy and could also endogenously respond to changes in tariffs. Since most MFN tariffs are ad valorem, the impact of dropping these observations for our sample size is small. For each country and HS pair, we take the minimum of the MFN and preferred rate, and then we take an unweighted average of the resulting tariffs across all HS codes in the NAICS-33 (manufacturing) sector and all years for which the tariffs are not missing.

We do several robustness tests for our tariff data: First, we use MFN tariffs since, at times, even if a preferred agreement is in place, firms trade using MFN rates since trading subject to preferred rates may require extra efforts for firms, e.g. additional forms to fill out. Second, we experiment with the weighting of HS goods and construct a weighted tariff for manufacturing that is weighted by the goods share in world trade in that year. Third, we use the information on applied rates, which is available for goods traded in positive quantities only. We assign the applied tariff to a good-country pair if available and otherwise assign the minimum of MFN and preferred rate. None of these alternative measures alters our quantitative results significantly.

OECD Enterprise Data We obtain information on the number of enterprises active in each country from the OECD Structural Statistics of Industry and Services for OECD and non-OECD countries. We also extract additional data from the National Statistical Agencies for Ukraine, Mexico, and Russia, for which the OECD data lacks information.

OECD Survival Statistics We also use the OECD Structural and Demographic Business Statistics database to obtain information on the 1-, 2-, 3-, 4-, and 5-year survival rates of manufacturing enterprises in many countries and years. The data is missing for some year and country combinations. We impute the missing survival rates by running a regression of log survival rates on the log of GDP, total employment, total population, and year and survival rate horizon fixed effects. We then use the estimated coefficients to predict the missing survival rates. In our baseline calibration, we use the 1-year survival rates since they correspond most closely to the idea of businesses that pay an entry cost to learn their productivity but then never end up producing positive quantities.

Penn World Tables We use the PWT 10.01 version of the Penn World Tables (see Feenstra, Inklaar, and Timmer 2015). We extract the total "expenditure side real GDP in chained PPPs in millions of 2017 dollars between 2003 and 2012. Likewise, we extract the total employment and total population of each country in each year.

C.2. Calibration

In this section, we provide more detail on the calibration of our model.

Bilateral Costs of Trade, MP, and Affiliates We estimate the following empirical gravity equation for each bilateral flow indexed by $x \in \{\tau_{\ell n}, \gamma_{i\ell}, \nu_{i\ell}\}$: trade flows, inward MP sales, and inward affiliate stocks.

$$y_{ij}^x = \exp\left(\alpha^x + \sum_{v \in \{d,\text{COL},\text{COM},\text{BOR}\}} \beta_v^x v_{ij} + \delta_x' X_{ij} + \varkappa_i + \zeta_j\right) + \epsilon_{ij}^x \tag{10}$$

The gravity variables, indexed by v, are the log distance d_{ij} between countries i and j, and dummies for colonial relations (COL), common language (COM) and common borders (BOR). The vector X_{ij} contains additional gravity controls, in particular bilateral tariffs and a free trade agreement dummy. The terms $\varkappa_i$ and ζ_j are origin and destination specific fixed effects.

Table 3 presents results for the gravity variables coefficients that we target in calibration, omitting untargeted coefficients for conciseness. We estimate equations via Poisson Pseudo Maximum Likelihood (Silva and Tenreyro 2006) as well as OLS, where the estimated elasticities are in line with those of similar regressions in Ramondo, Rodríguez-Clare, and Tintelnot (2015). In computing manufacturing tariffs, it is necessary to decide whether to use raw averages of the tariffs of all goods in manufacturing or weight in some way. For robustness, we report results using both unweighted and weighted tariffs.

We specify the following functional forms of the bilateral trade, MP, and fixed costs:

$$\begin{aligned} \log \tau_{\ell n} &= \overline{\tau}_n \times \mathbb{1}[\ell \neq n] + \sum_{v \in \{d,\text{COL},\text{COM},\text{BOR}\}} \kappa_\tau^v v_{\ell n} + \log(1 + t_{\ell n}) \\ \log \gamma_{i\ell} &= \overline{\gamma}_\ell \times \mathbb{1}[i \neq \ell] + \sum_{v \in \{d,\text{COL},\text{COM},\text{BOR}\}} \kappa_\gamma^v v_{i\ell} \\ \log \nu_{i\ell} &= \overline{\nu}_\ell \times \mathbb{1}[i \neq \ell] + \sum_{v \in \{d,\text{COL},\text{COM},\text{BOR}\}} \kappa_\nu^v \log v_{i\ell} \end{aligned} \tag{11}$$

where v indexes the same set of gravity variables as in the regression (10). The $\{\overline{\tau}_n, \overline{\gamma}_\ell, \overline{\nu}_\ell\}$ components represent the costs of doing an activity across borders versus within borders. In our calibration procedure, we estimate the destination-specific components $\{\overline{\tau}_n, \overline{\gamma}_\ell, \overline{\nu}_\ell\}$ by targeting the own-shares of each activity, and the gravity-variable-specific elasticities by targeting the estimated coefficients on the corresponding gravity variables in Table 3a using unweighted tariffs. Table 7a reports the estimated elasticities.

Figure 7b shows histograms of our calibrated trade costs, MP costs, and the bilateral component of fixed costs; we exclude the diagonal entries of all cost matrices from the histogram since they are normalized to 1. Our estimated trade costs are substantial, similar to prior estimates from studies featuring trade and multinational production (e.g. Ramondo and Rodríguez-Clare 2013).

In contrast, our estimated MP costs are small compared to previous studies such as Ramondo and Rodríguez-Clare (2013) or Arkolakis, Ramondo, et al. (2018). These differences arise from the fact that we allow for fixed costs in addition to MP cost. We use affiliate count data to separate fixed costs from MP cost, while previous studies that only use MP sales and trade data cannot separate these two costs. In the data, there are few MP affiliates, but they account for a large sales volume in their host countries; to match these patterns, we estimate large fixed costs and therefore smaller MP costs. The presence of economically significant fixed costs is in line with Hjort, Malmberg, and Schoellman (2022), which finds that, in real terms, labor compensation of middle management is both an important component of the cost of doing multinational business abroad and also does not vary much across MNE locations.

In the Online Appendix, we regress our estimates of the MP costs and the bilateral component of the fixed costs on our set of gravity variables to understand their determinants. For the bilateral component of fixed costs, we find a large positive coefficient on language, consistent with evidence of language barriers in foreign MNE activity by Guillouët et al. (2024). Our cost estimates are also consistent with the finding in Alviarez, Cravino, and Ramondo (2023) that within-destination market firm market shares for manufacturing decline relatively little with distance. In particular, MP costs, which are the main determinant of within-destination market shares in our model, react little to distance in our calibrations.

Model Fit Figure 8 presents our calibrated model's performance on an important untargeted moment: the US sales premium of multinational firms based in the US. We calculate the average sales among groups of firms with presence in different minimum numbers of foreign locations, normalized by the average sales of non-MNE firms based in the US. Figure 8 shows both these MNE sales premia computed in our calibrated model and the empirical premia documented by Antràs et al. (2024b). The model premia closely mirror the empirical premia, reflecting that more productive firms broadly have both more foreign affiliate locations and higher sales. For example, MNEs (any firm conducting production in at least two countries) are more than 40 times larger than non-MNEs in the US in both the model and the (untargeted) data.

	Unweighted Tariffs			Weighted Tariffs		
	Trade (1)	MP (2)	Affiliates (3)	Trade (4)	MP (5)	Affiliates (6)
Log Distance	-0.687***	-0.289**	-0.681***	-0.689***	-0.284**	-0.681***
	(0.0541)	(0.106)	(0.0847)	(0.0540)	(0.107)	(0.0851)
Colony	0.0776	-0.00471	0.232	0.0758	0.000476	0.260
	(0.125)	(0.131)	(0.144)	(0.124)	(0.131)	(0.146)
Contiguity	0.448***	0.424**	0.448***	0.440***	0.412*	0.436***
	(0.0697)	(0.164)	(0.0979)	(0.0697)	(0.163)	(0.0983)
Language	0.152	0.468**	0.561***	0.160	0.476**	0.577***
	(0.102)	(0.145)	(0.155)	(0.101)	(0.146)	(0.161)
Observations	992	707	710	992	707	710

(A) Estimation with PPML

	Unweighted Tariffs			Weighted Tariffs		
	Trade (1)	MP (2)	Affiliates (3)	Trade (4)	MP (5)	Affiliates (6)
Log Distance	-1.106***	-0.822***	-0.800***	-1.104***	-0.814***	-0.796***
	(0.0510)	(0.122)	(0.0740)	(0.0509)	(0.122)	(0.0741)
Colony	0.763***	0.940***	0.659***	0.761***	0.954***	0.669***
	(0.108)	(0.244)	(0.149)	(0.108)	(0.244)	(0.149)
Contiguity	0.325***	0.734***	0.410***	0.325***	0.732***	0.408***
	(0.0913)	(0.195)	(0.118)	(0.0912)	(0.195)	(0.118)
Language	0.00442	0.559*	0.183	0.0000334	0.570*	0.190
	(0.134)	(0.284)	(0.173)	(0.133)	(0.284)	(0.173)
Observations	992	707	710	992	707	710

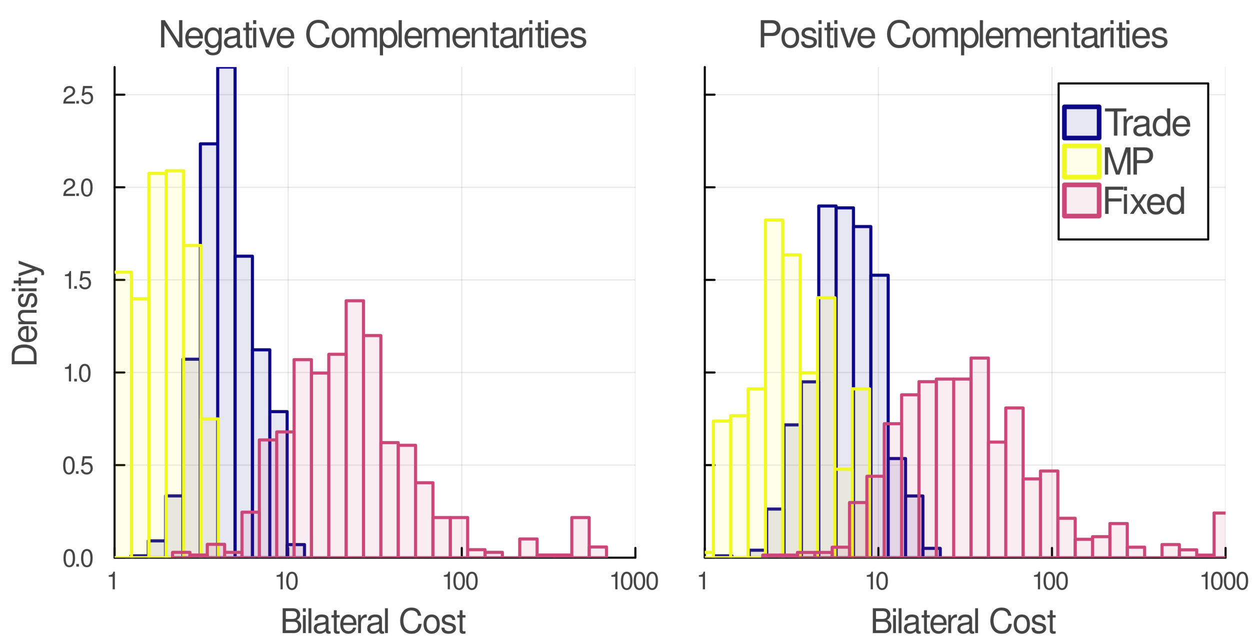
(B) Estimation with OLS

TABLE 3: TRADE, MP, AND FOREIGN AFFILIATE GRAVITY IN THE DATA

The table presents the estimated coefficients from estimating gravity equations. The outcome variable differs across the columns: bilateral manufacturing trade flows, bilateral multinational production sales, and bilateral foreign affiliate stocks. The standard gravity controls serve as explanatory variables. All estimating equations also include origin and destination fixed effects and additional controls for bilateral tariffs and a regional trade agreement dummy. Tariffs are averages across all manufacturing goods, either unweighted or weighted by the global trade shares of each good. The specifications exclude the diagonal entries of the respective flow matrix. Robust standard errors are in parentheses. We denote different levels of significance as follows: *** Significant at 1 percent level, ** Significant at 5 percent level, and * Significant at 10 percent level.

	Negative Complementarities			Positive Complementarities		
	Trade	MP	Affiliates	Trade	MP	Affiliates
Language	0.047	0.066	0.080	0.064	0.106	0.197
Contiguity	0.143	0.060	0.053	0.162	0.134	-0.010
Colony	0.025	-0.028	0.187	0.025	-0.039	0.207
Log Distance	0.215	0.001	0.346	0.270	0.016	0.429

(A) Estimated Cost Elasticities of the Gravity Variables



(B) Distribution of Calibrated Bilateral Costs

FIGURE 7: TRADE COSTS, MP COSTS, AND THE BILATERAL COMPONENT OF FIXED COSTS

This figure summarizes the estimated bilateral costs. Figure 7b shows a histogram of the three bilateral cost matrices in the model: trade costs, MP costs, and the bilateral component of fixed costs. We omit the own-country costs which are normalized to 1 for all three types of costs. For MP and the bilateral component of fixed costs, we also omit country pairs where MP is zero, since we set the MP costs to be infinity in those cases. Table 7a presents the calibrated elasticities of all gravity variables in each of the bilateral costs in the model as specified in equation (11).

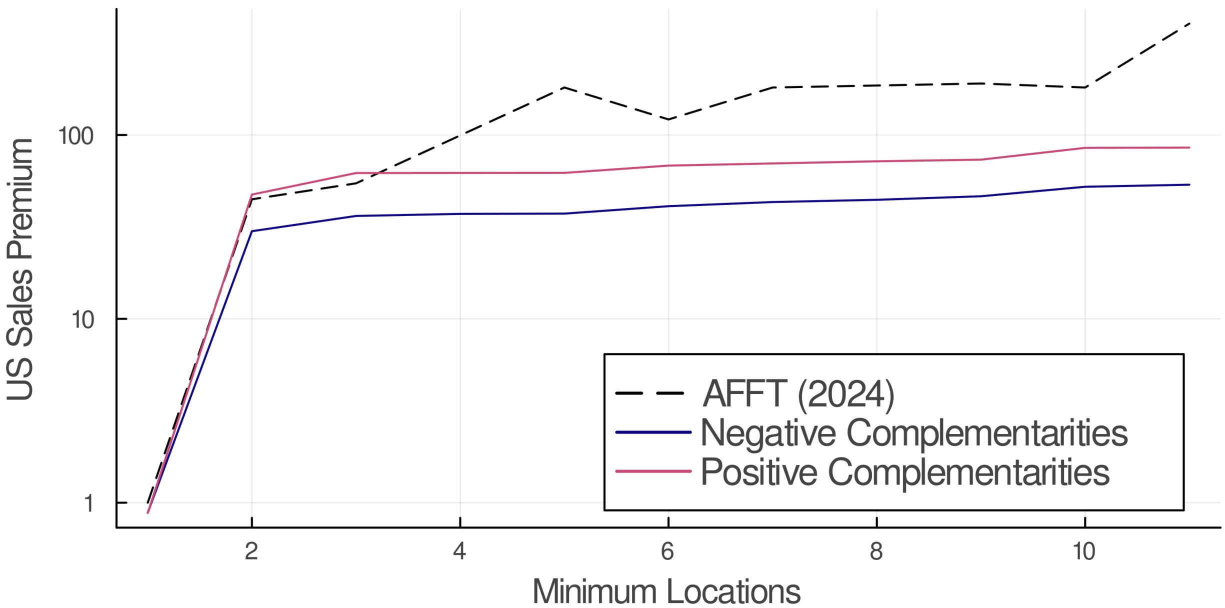


FIGURE 8: MULTINATIONAL SALES PREMIA AND NUMBER OF FOREIGN AFFILIATES IN THE DATA AND THE BASELINE CALIBRATION

The figure compares size sales premia in the model and in the US data obtained from Antràs et al. (2024b). The sales premium is measured as the relative sales of US-based multinational firms compared to non-MNEs.

SUPPLEMENTAL ONLINE APPENDIX

Combinatorial Discrete Choice: Theory and Application to Multinational Production

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D. The Mathematics of CDCPs

D.1. Definitions and Existing Results

Definition (Poset)**.** A poset (partially-ordered set) $(P, \leq)$ is a set together with a partial ordering that is:

1. reflexive: for all $x \in P$, $x \leq x$;
2. antisymmetric: for any pair with $x \leq y$ and $y \leq x$, it must be that $x = y$; and
3. transitive: for any elements with $x \leq y$ and $y \leq z$, it must be that $x \leq z$.

The dual poset $(P, \leq_D)$ is the set together with the dual ordering that is define $x \leq_D y$ iff $y \leq x$.

Definition (Lattice)**.** A lattice is a poset $(L, \leq)$ where, for any $x, y \in L$, there is a unique supremum $\sup \{x, y\}$ and infimum $\inf \{x, y\}$ with respect to $\leq$. The lattice is complete if, for any subset $S \subseteq L$, there is a unique supremum $\sup S$ and infimum $\inf S$. A sublattice $(L', \leq)$ is a subset of points $L' \subseteq L$ that is itself a lattice.

Definition (Directed complete)**.** A poset $(P, \leq)$ is direct complete if, for all subsets $D \subseteq P$ that is closed under pairwise supremum, $\sup D$ exists.

Definition (Scott continuity)**.** A function between two posets $f : (P, \leq_P) \to (Q, \leq_Q)$ is Scott-continuous if, for every subset $D \subseteq P$ that is closed under pairwise supremum, the image of the supremum is the supremum of the image: $f (\sup \{D\}) = \sup \{f (x) \mid x \in D\}$.

Definition (Order-preserving (reversing))**.** A mapping $\Phi : P \to P$ is order-preserving if, given $x < y$, $\Phi (x) \leq \Phi (y)$. It is order-reversing if, given $x < y$, $\Phi (y) \leq \Phi (x)$. If the mapping is either order-preserving or order-reversing, it is monotonic.

Theorem (Tarski (1955))**.** *Given a complete lattice $(L, \leq)$ and an order-preserving endomap $f : L \to L$, the set of fixed points of f forms a complete lattice.*

Theorem (Kleene)**.** *Given a directed-complete partial order $(D, \leq)$ with a least element x and Scott-continuous endomap $f : D \to D$, f has a least fixed point, which is $\sup \{f^n (x) \mid n \in \mathbb{N}\}$.*

Theorem (Klimeš (1981))**.** *Given a complete lattice $(L, \leq)$ and an order-reversing endomap $f : L \to L$. Then, there is a least element u of L so that $(u, f (u))$ is a fixed edge of f. There is similarly a greatest element v with $(v, f (v))$ a fixed edge of f. Moreover, $v = f (u)$.*

D.2. Proofs of Main Results

The squeezing step We first show that SCD-C from above and below are necessary and sufficient for Φ to be order-reversing and order-preserving, respectively.

Proposition 1. *The mapping Φ defined in Definition 4 is*

1. *order-reversing iff the underlying objective function f obeys SCD-C from above; and*
2. *order-preserving iff the underlying objective function f obeys SCD-C from below.*

Proof. We show the first statement. Let $\mathcal{L} \subset \mathcal{L}'$ be two arbitrary nested decision sets.

Start with the converse. Suppose f obeys SCD-C from above. If $\Phi(\mathcal{L}')$ is empty, then it is contained in $\Phi(\mathcal{L})$ trivially; so let $\ell \in \Phi(\mathcal{L}')$ be an arbitrary element. Then, by definition of Φ, $D_\ell f(\mathcal{L}) \geq 0$. With SCD-C from above, it must be that $D_\ell f(\mathcal{L}') \geq 0$; hence, $\ell \in \Phi(\mathcal{L})$. Then, $\Phi(\mathcal{L}') \subseteq \Phi(\mathcal{L})$ and Φ is order-reversing.

Now consider the forward direction. Let ℓ be an arbitrary element so that $D_\ell f(\mathcal{L}') \geq 0$. If no such element exists, then SCD-C from above holds vacuously, so suppose at least one such ℓ exists. Then, by definition, $\ell \in \Phi(\mathcal{L}') \subseteq \Phi(\mathcal{L})$ since Φ is order-reversing. Then, by definition of Φ, it must be that $D_\ell f(\mathcal{L}) \geq 0$.

A reverse argument holds for SCD-C from below. $\square$

Corollary 1. *Consider the objective function $f : \mathcal{P}(L) \to \mathbb{R}$.*

1. *Quasisupermodularity of f is sufficient for SCD-C from below; quasisubmodularity is sufficient for SCD-from above.*
2. *If L is finite, the function exhibits increasing marginal values iff it is supermodular and decreasing marginal values iff it is submodular.*

Proof. We show the statements for quasisubmodularity, submodularity, and SCD-C from below. Similar arguments follow for quasisupermodularity, supermodularity, and SCD-C from above.

1. Suppose f satisfies quasisubmodularity: i.e. for all $x, y \in \mathcal{P}(L)$,

$$\begin{aligned} f(x \cup y) \geq f(y) \quad &\Rightarrow \quad f(x) \geq f(x \cap y) \\ f(x \cup y) > f(y) \quad &\Rightarrow \quad f(x) > f(x \cap y) \end{aligned}$$

and let $\mathcal{L} \in \mathcal{P}(L)$, $\ell \in L$ with $D_\ell f(\mathcal{L}) \geq 0$. Select any $\mathcal{L}' \subseteq \mathcal{L}$. We show that $D_\ell(\mathcal{L}') \geq 0$. Let $\mathcal{J} \equiv \mathcal{L}' \cup \{\ell\}$ and $\mathcal{K} \equiv \mathcal{L} \setminus \{\ell\}$. Then,

$$\begin{aligned} D_\ell f(\mathcal{L}) &= f(\mathcal{L} \cup \{\ell\}) - f(\mathcal{L} \setminus \{\ell\}) \\ &= f(\mathcal{J} \cup \mathcal{K}) - f(\mathcal{K}) \geq 0 \\ \Rightarrow f(\mathcal{J}) &\geq f(\mathcal{J} \cap \mathcal{K}) \end{aligned}$$

where the last line follows from quasisupermodularity. Then, it immediately follows that $D_\ell(\mathcal{L}') \geq 0$.

2. Suppose f has decreasing marginal values. We show that f is submodular. Let x and y be arbitrary elements of $\mathcal{P}(L)$. Let $\mathcal{L} \equiv x \cap y$, $\mathcal{J} \equiv y \setminus x$, and $\mathcal{K} \equiv y \setminus x$. Note that the sets are disjoint and

$$\begin{aligned} & f(x \cup y) + f(x \cap y) - f(x) - f(y) \\ & = [f(\mathcal{K} \cup \mathcal{L} \cup \mathcal{J}) - f(\mathcal{K} \cup \mathcal{L})] - [f(\mathcal{L} \cup \mathcal{J}) - f(\mathcal{L})] \end{aligned}$$

so it suffices to show this difference is non-negative. We proceed with induction on $|\mathcal{J}|$. Suppose $|\mathcal{J}| = 1$. WLOG, let $\mathcal{J} \equiv \{\ell\}$. Then,

$$\begin{aligned} & [f(\mathcal{K} \cup \mathcal{L} \cup \{\ell\}) - f(\mathcal{K} \cup \mathcal{L})] - [f(\mathcal{L} \cup \{\ell\}) - f(\mathcal{L})] \\ & = D_\ell(\mathcal{K} \cup \mathcal{L}) - D_\ell(\mathcal{L}) \end{aligned}$$

since $\ell \notin \{\mathcal{K} \cup \mathcal{L}\}$. By increasing marginal values, we establish this difference is positive in the base case. Now suppose increasing marginal values implies supermodularity as long as $|\mathcal{J}| = k$. Consider the case where $|\mathcal{J}| = k+1$. Select an element $\ell \in \mathcal{J}$ and define $\tilde{\mathcal{J}} \equiv \mathcal{J} \setminus \{\ell\}$ so that $\left|\tilde{\mathcal{J}}\right| = k$. Then,

$$\begin{aligned} & \left[f\left(\mathcal{K} \cup \mathcal{L} \cup \tilde{\mathcal{J}} \cup \{\ell\}\right) - f(\mathcal{K} \cup \mathcal{L})\right] - \left[f\left(\mathcal{L} \cup \tilde{\mathcal{J}} \cup \{\ell\}\right) - f(\mathcal{L})\right] \\ = \; & \left[f\left(\mathcal{K} \cup \mathcal{L} \cup \tilde{\mathcal{J}} \cup \{\ell\}\right) - f\left(\mathcal{K} \cup \mathcal{L} \cup \tilde{\mathcal{J}}\right) + f\left(\mathcal{K} \cup \mathcal{L} \cup \tilde{\mathcal{J}}\right) - f(\mathcal{K} \cup \mathcal{L})\right] \\ & - \left[f\left(\mathcal{L} \cup \tilde{\mathcal{J}} \cup \{\ell\}\right) - f\left(\mathcal{L} \cup \tilde{\mathcal{J}}\right) + f\left(\mathcal{L} \cup \tilde{\mathcal{J}}\right) - f(\mathcal{L})\right] \\ = \; & \left[f\left(\mathcal{K} \cup \mathcal{L} \cup \tilde{\mathcal{J}} \cup \{\ell\}\right) - f\left(\mathcal{K} \cup \mathcal{L} \cup \tilde{\mathcal{J}}\right)\right] - \left[f\left(\mathcal{L} \cup \tilde{\mathcal{J}} \cup \{\ell\}\right) - f\left(\mathcal{L} \cup \tilde{\mathcal{J}}\right)\right] \\ & + \left[f\left(\mathcal{K} \cup \mathcal{L} \cup \tilde{\mathcal{J}}\right) - f(\mathcal{K} \cup \mathcal{L})\right] - \left[f\left(\mathcal{L} \cup \tilde{\mathcal{J}}\right) - f(\mathcal{L})\right] \end{aligned}$$

where the first line is positive by increasing marginal values and the second line is positive by the inductive assumption.

□

We now formally prove Theorem 1.

Proof. First, note that

$$\left[\underline{\mathcal{L}}^{(k+1)}, \overline{\mathcal{L}}^{(k+1)}\right] = \begin{cases} \left[\Phi\left(\overline{\mathcal{L}}^{(k)}\right), \Phi\left(\underline{\mathcal{L}}^{(k)}\right)\right] & \text{if } f \text{ satisfies SCD-C from above} \\ \left[\Phi\left(\underline{\mathcal{L}}^{(k)}\right), \Phi\left(\overline{\mathcal{L}}^{(k)}\right)\right] & \text{if } f \text{ satisfies SCD-C from below} \end{cases}$$

and thus as long as $\underline{\mathcal{L}}^{(k)} \subseteq \mathcal{L}^\star \subseteq \overline{\mathcal{L}}^{(k)}$, then the monotonicity of Φ guarantees $\underline{\mathcal{L}}^{(k+1)} \subseteq \mathcal{L}^\star \subseteq \overline{\mathcal{L}}^{(k+1)}$ because $\Phi(\mathcal{L}^\star) = \mathcal{L}^\star$.

We now show that the bounding pair (weakly) tightens each iteration using induction on the iteration. Start with $k = 1$. Then, $\varnothing = \underline{\mathcal{L}}^{(0)} \subseteq \underline{\mathcal{L}}^{(1)} \subseteq \overline{\mathcal{L}}^{(1)} \subseteq \mathcal{L}^{(0)} = L$ trivially. Then, assume $\underline{\mathcal{L}}^{(k-1)} \subseteq \underline{\mathcal{L}}^{(k)} \subseteq \overline{\mathcal{L}}^{(k)} \subseteq \overline{\mathcal{L}}^{(k-1)}$. Applying Φ to each decision set and using monotonicity,

$$\begin{cases} \Phi\left(\overline{\mathcal{L}}^{(k-1)}\right) \subseteq \Phi\left(\overline{\mathcal{L}}^{(k)}\right) \subseteq \Phi\left(\underline{\mathcal{L}}^{(k)}\right) \subseteq \Phi\left(\underline{\mathcal{L}}^{(k-1)}\right) & \text{if } f \text{ satisfies SCD-C from above} \\ \Phi\left(\underline{\mathcal{L}}^{(k-1)}\right) \subseteq \Phi\left(\underline{\mathcal{L}}^{(k)}\right) \subseteq \Phi\left(\overline{\mathcal{L}}^{(k)}\right) \subseteq \Phi\left(\overline{\mathcal{L}}^{(k-1)}\right) & \text{if } f \text{ satisfies SCD-C from below} \end{cases}$$
$$\Rightarrow \underline{\mathcal{L}}^{(k)} \subseteq \underline{\mathcal{L}}^{(k+1)} \subseteq \overline{\mathcal{L}}^{(k+1)} \subseteq \overline{\mathcal{L}}^{(k)}$$

from the definition of S.

Finally, each iteration of the squeezing step must add at least one additional item to the lower bounding set or exclude at least one additional item from the upper bounding set which can occur a maximum of $|L|$ times. □

Corollary 2. *As long as f satisfies SCD-C from above or below, the squeezing step S itself is an increasing mapping.*

Proof. Define the partial order $\leq_{[]}$ over the set of bounding pairs so that $\left[\underline{\mathcal{L}}, \overline{\mathcal{L}}\right] \leq_{[]} \left[\underline{\mathcal{L}}', \overline{\mathcal{L}}'\right]$ iff both $\underline{\mathcal{L}} \subseteq \underline{\mathcal{L}}'$ and $\overline{\mathcal{L}}' \subseteq \overline{\mathcal{L}}$, i.e. order the bounding pairs by tightness. We prove the corollary for the case of SCD-C from above.

Let $\left[\underline{\mathcal{L}}, \overline{\mathcal{L}}\right] \leq_{[]} \left[\underline{\mathcal{L}}', \overline{\mathcal{L}}'\right]$ be two arbitrary ordered bounding pairs. Since Φ is order-reversing, $S\left(\left[\underline{\mathcal{L}}, \overline{\mathcal{L}}\right]\right) = \left[\Phi\left(\overline{\mathcal{L}}\right), \Phi\left(\underline{\mathcal{L}}\right)\right]$ is a bounding pair and similarly for $\left[\underline{\mathcal{L}}', \overline{\mathcal{L}}'\right]$. Since $\overline{\mathcal{L}}' \subseteq \overline{\mathcal{L}}$ and Φ order-reversing, we have that $\Phi\left(\overline{\mathcal{L}}\right) \subseteq \Phi\left(\overline{\mathcal{L}}'\right)$. Similarly, $\underline{\mathcal{L}} \subseteq \underline{\mathcal{L}}'$ implies that $\Phi\left(\underline{\mathcal{L}}'\right) \subseteq \Phi\left(\underline{\mathcal{L}}\right)$. Thus,

$$\left[\Phi\left(\overline{\mathcal{L}}\right), \Phi\left(\underline{\mathcal{L}}\right)\right] \leq_{[]} \left[\Phi\left(\overline{\mathcal{L}}'\right), \Phi\left(\underline{\mathcal{L}}'\right)\right]$$

which completes the proof.

Note that a bounding pair $\left[\underline{\mathcal{L}}, \overline{\mathcal{L}}\right]$ defines an implicit sublattice on $\left(\mathcal{P}\left(L\right), \subseteq\right)$. Thus, letting $\mathcal{B}$ be the set of sublattices of $\left(\mathcal{P}\left(L\right), \subseteq\right)$, then S is an increasing map on the lattice $\left(\mathcal{B}, \supseteq\right)$. □

Corollary 3. *When f satisfies SCD-C, so that $\left[\underline{\mathcal{L}}^{(K)}, \overline{\mathcal{L}}^{(K)}\right] = S^K\left(\left[\varnothing, L\right]\right)$ is a fixed point by Theorem 1, then*

1. if f satisfies SCD-C from below, then $\underline{\mathcal{L}}^{(K)}$ and $\overline{\mathcal{L}}^{(K)}$are the smallest and largest fixed points of Φ respectively; and
2. if f satisfies SCD-C from above, then $\underline{\mathcal{L}}^{(K)}$ and $\overline{\mathcal{L}}^{(K)}$are the smallest and largest fixed edges of Φ respectively.

Proof. We show each statement in turn. Let $D \subseteq \mathcal{P}(L)$ be closed under pairwise join and denote $\Phi(D) \equiv \{\Phi(\mathcal{L}) \mid \mathcal{L} \in D\}$ the image of D.

1. In this case, Φ is order-preserving. We first show that Φ is Scott-continuous. Let $x = \sup D$ which exists in $\mathcal{P}(L)$ since it is a complete lattice. Because Φ is order preserving, we have that for all $\mathcal{L} \in D$, $\Phi(\mathcal{L}) \subseteq \Phi(\sup D) = \Phi(x)$ so the image of the supremum is an upper bound. Additionally, since D is finite, it is closed under join:

$$\bigcap \{\mathcal{L} \mid \mathcal{L} \in D\} \in D$$

and thus closed under supremum. In other words, $x \in D$. Then, $\Phi(x) \in \Phi(D)$ so $\Phi(x)$ is the least upper bound of $\Phi(D)$. Hence, Φ is Scott-continuous. With a direct application of Kleene's fixed point theorem, $\underline{\mathcal{L}}^{(K)} = \Phi^K(\emptyset)$ is the smallest fixed point of Φ. Now consider the dual poset $(\mathcal{P}(L), \subseteq_D) = (\mathcal{P}(L), \supseteq)$. Note that the least element in the dual poset is L, since $L \supseteq \mathcal{L}$ for all $\mathcal{L} \in \mathcal{P}(L)$. Define Φ_D as an endomap on the dual poset with $\Phi_D(\mathcal{L}) \equiv \Phi(\mathcal{L})$ and note that Φ is still order-preserving: for all $\mathcal{L} \supset \mathcal{L}'$, we have $\Phi(\mathcal{L}) \supseteq \Phi(\mathcal{L}')$. By an identical argument as above, it is thus Scott-continuous, so by direct application of Kleene's fixed point theorem, $\overline{\mathcal{L}}^{(K)} = \Phi_D^K(L)$ is the smallest fixed point on Φ_D. It is thus the largest fixed point of Φ.
2. Define
$$E \equiv \left\{\mathcal{L} \mid \exists \mathcal{L}' \in \mathcal{P}(L) \text{ where } \mathcal{L} = \Phi\left(\mathcal{L}'\right) \text{ and } \mathcal{L}' = \Phi(\mathcal{L})\right\}$$
as the set of fixed edges of Φ. We first show $E = \left\{\mathcal{L} \mid \Phi^2(\mathcal{L}) = \mathcal{L}\right\}$ the set of fixed points of Φ^2. First, let $\Phi^2(\mathcal{L}) = \mathcal{L}$ e an arbitrary fixed point of Φ^2. Then, $\Phi(\Phi(\mathcal{L})) = \mathcal{L}$ so $\mathcal{L} \in E$ with $\mathcal{L}' = \Phi(\mathcal{L})$. Now let $\mathcal{L} \in E$. Then, $\Phi(\Phi(\mathcal{L})) = \Phi(\mathcal{L}') = \mathcal{L}$ so $\mathcal{L}$ is a fixed point of Φ^2. The two sets are equivalent. Because Φ^2 is order-preserving in this case, by identical arguments we have that $\underline{\mathcal{L}}^{(K)} = \Phi^{2K}(\emptyset)$ is the smallest fixed point of Φ^2, i.e. the smallest element of E or the smallest fixed edge of Φ. Then, $\overline{\mathcal{L}}^{(K)} = \sup\left\{\Phi\left(\underline{\mathcal{L}}^{(K)}\right), \Phi\left(\overline{\mathcal{L}}^{(K)}\right)\right\} = \Phi\left(\underline{\mathcal{L}}^{(K)}\right)$ since Φ is order-reversing, which implies that $\overline{\mathcal{L}}^{(K)} \in E$. We show it is the largest element of E to complete the proof. Let $\mathcal{L} \in E$ be an arbitrary element. Let $\mathcal{L}^{\sup} \in E$ be the greatest element. Then, with $\overline{\mathcal{L}}^{(K)} \subseteq \mathcal{L}^{\sup}$. Define $\mathcal{L}' = \Phi(\mathcal{L}^{\sup})$ and note that it is also in E. Then, $\mathcal{L}' = \Phi(\mathcal{L}^{\sup}) \subseteq \Phi\left(\overline{\mathcal{L}}^{(K)}\right) = \underline{\mathcal{L}}^{(K)}$ since Φ is order-reversing. Then, $\mathcal{L}' = \underline{\mathcal{L}}^{(K)}$ since it is the smallest element, so $\Phi(\mathcal{L}') = \mathcal{L} = \overline{\mathcal{L}}^{(K)}$ so it is the greatest element.

□

The branching step We show that the branching procedure identifies fixed points of the mapping Φ.

Theorem 3. *The branching procedure identifies precisely all fixed points of Φ within $\left[\underline{\mathcal{L}}, \overline{\mathcal{L}}\right]$.*

Proof. Consider $\Phi : \mathcal{P}(L) \rightarrow \mathcal{P}(L)$. Suppose $\left[\underline{\mathcal{L}}, \overline{\mathcal{L}}\right]$ is a fixed point (edge) of order-preserving (reversing) Φ. We prove by induction on the cardinality of $\overline{\mathcal{L}} \setminus \underline{\mathcal{L}}$.

Suppose $\overline{\mathcal{L}} \setminus \underline{\mathcal{L}} = \{\ell\}$ so that the cardinality is 1. Then, $\overline{\mathcal{L}} = \underline{\mathcal{L}} \cup \{\ell\}$ so there are no other sets sandwiched by $\left[\underline{\mathcal{L}}, \overline{\mathcal{L}}\right]$. Since the bounding pair is itself a fixed point (edge), the base case holds vacuously by branching 0 times. Now, suppose branching identifies all fixed points if there are k elements in $\overline{\mathcal{L}} \setminus \underline{\mathcal{L}}$. Consider a scenario where $\overline{\mathcal{L}} \setminus \underline{\mathcal{L}}$ contains $k+1$ elements. Branching necessarily partitions the remaining search space into two smaller problems: the one defined by $\left[\underline{\mathcal{L}} \cup \{\ell\}, \overline{\mathcal{L}}\right]$ and the one defined by $\left[\underline{\mathcal{L}}, \overline{\mathcal{L}} \setminus \{\ell\}\right]$. Any fixed point contained in exactly one of these cells. Then, by inductive assumption, branching identifies all fixed points in each cell separately, and thus all fixed points of the larger problem. □

Corollary 4. *The branching procedure identifies precisely all fixed points of Φ if applied to $\left[\underline{\mathcal{L}}, \overline{\mathcal{L}}\right] = S^K([\emptyset, L])$.*

Proof. By Theorem 1, $\left[\underline{\mathcal{L}}, \overline{\mathcal{L}}\right]$ are the smallest and largest fixed points (edges) of Φ. Thus, any other fixed point (edge) must be sandwiched by them. The corollary follows immediately from Theorem 3. □

Corollary. *Suppose f exhibits SCD-C from above and $\left[\underline{\mathcal{L}}, \overline{\mathcal{L}}\right]$ is the converged pair from squeezing where $\left|\overline{\mathcal{L}} \setminus \underline{\mathcal{L}}\right| > 2$. Then, the number of decision sets left after the branching procedure completes, $\left|\Lambda_f\left(\underline{\mathcal{L}}, \overline{\mathcal{L}}\right)\right|$, is strictly less than the number of decisions to evaluate with brute force.*

Proof. Since f exhibits SCD-C from above, Φ is order-reversing. Then, note that if $\mathcal{L}_1 \neq \mathcal{L}_2$ are both fixed points of Φ, it cannot be that one is nested in the other. Without loss of generality, if it were the case that $\mathcal{L}_1 \subset \mathcal{L}_2$, then $\mathcal{L}_2 = \Phi(\mathcal{L}_2) \subseteq \Phi(\mathcal{L}_1) = \overline{\mathcal{L}}_1$, a contradiction.

Moreover, since Φ is order reversing, the bounding pair of sets $\underline{\mathcal{L}}$ and $\overline{\mathcal{L}}$ are each fixed edges but not fixed points; thus, they are not the optimal decision set and can be disregarded when approaching the remaining problem with brute force. Brute force thus must choose among $\left|\mathcal{P}\left(\overline{\mathcal{L}} \setminus \underline{\mathcal{L}}\right)\right| - 2$ decision sets.

For a contradiction, suppose that $\left|\Lambda\left(\left[\underline{\mathcal{L}}, \overline{\mathcal{L}}\right]\right)\right| = \left|\mathcal{P}\left(\overline{\mathcal{L}} \setminus \underline{\mathcal{L}}\right)\right| - 2$. Then, it must be that

$$\Lambda\left(\left[\underline{\mathcal{L}}, \overline{\mathcal{L}}\right]\right) = \left\{\underline{\mathcal{L}} \cup \mathcal{L} \mid \mathcal{L} \in \mathcal{P}\left(\overline{\mathcal{L}} \setminus \underline{\mathcal{L}}\right)\right\} \setminus \left\{\underline{\mathcal{L}}, \overline{\mathcal{L}}\right\}$$

since L is finite. Because $\left|\overline{\mathcal{L}} \setminus \underline{\mathcal{L}}\right| > 2$, we can select three distinct items from $\overline{\mathcal{L}} \setminus \underline{\mathcal{L}}$ and let them be ℓ_1, ℓ_2, and ℓ_3. Note then that

$$\underline{\mathcal{L}} \subset \left(\underline{\mathcal{L}} \cup \{\ell_1\}\right) \subset \left(\underline{\mathcal{L}} \cup \{\ell_1, \ell_2\}\right) \subset \overline{\mathcal{L}}$$

and hence both are in $\Lambda\left(\left[\underline{\mathcal{L}}, \overline{\mathcal{L}}\right]\right)$. Then, there are two fixed points of Φ where one is contained in the other, a contradiction. $\square$

The generalized squeezing step We first establish the following result, to write the generalized squeezing step more simply.

Lemma. *Suppose the underlying objective function f satisfies SCD-T and let $\ell \in L$ and $\mathcal{L}, \mathcal{L}' \in \mathcal{P}(L)$ where $\mathcal{L} \subset \mathcal{L}'$.*

1. *If the underlying objective function f also satisfies SCD-C from above, then $z_\ell^g(\mathcal{L}) \geq z_\ell^g(\mathcal{L}')$.*
2. *If the underlying objective function f also satisfies SCD-C from below, then $z_\ell^g(\mathcal{L}) \geq z_\ell^g(\mathcal{L}')$.*

Proof. Begin with SCD-C from above. Then,

$$0 = D_\ell\left(\mathcal{L}, z_\ell^g(\mathcal{L})\right) \qquad \Rightarrow \qquad 0 \leq D_\ell\left(\mathcal{L}', z_\ell^g(\mathcal{L})\right) \qquad \Rightarrow \qquad z_\ell^g\left(\mathcal{L}'\right) \geq z_\ell^g(\mathcal{L})$$

where the first implication follows from SCD-C from above and the second from SCD-T. Likewise, suppose f instead satisfies SCD-T from below. Then,

$$0 = D_\ell\left(\mathcal{L}, z_\ell^g(\mathcal{L})\right) \qquad \Rightarrow \qquad 0 \geq D_\ell\left(\mathcal{L}', z_\ell^g(\mathcal{L})\right) \qquad \Rightarrow \qquad z_\ell^g\left(\mathcal{L}'\right) \leq z_\ell^g(\mathcal{L})$$

which completes the proof. $\square$

Thus, similarly to the squeezing step, the generalized squeezing step applied to a pair of bounding set functions simplifies to

$$S^g\left(\left[\underline{\mathcal{L}}(\cdot), \overline{\mathcal{L}}'(\cdot)\right]\right) \equiv \begin{cases} \left[\Phi^g\left(\overline{\mathcal{L}}(\cdot), \cdot\right), \Phi^g\left(\underline{\mathcal{L}}(\cdot), \cdot\right)\right] & \text{if } f \text{satisfies SCD-C from above} \\ \left[\Phi^g\left(\underline{\mathcal{L}}(\cdot), \cdot\right), \Phi^g\left(\overline{\mathcal{L}}(\cdot), \cdot\right)\right] & \text{if } f \text{satisfies SCD-C from below} \end{cases}$$

if the objective function also satisfies SCD-T. We now prove Theorem 2.

Proof. We prove the case of SCD-C from above. Let $\Phi(\mathcal{L}, z) \equiv \{\ell \mid D_\ell f(\mathcal{L}, z) \geq 0\}$ be the mapping Φ evaluated at the type z. Consider a pair of bounding set functions $\left[\underline{\mathcal{L}}(\cdot), \overline{\mathcal{L}}(\cdot)\right]$ and the partition it induces $\mathcal{T}\left(\left[\underline{\mathcal{L}}(\cdot), \overline{\mathcal{L}}(\cdot)\right]\right)$. Select an arbitrary interval from the partition $\mathcal{Z}_t$; let its bounding pair be $\left[\underline{\mathcal{L}}_t, \overline{\mathcal{L}}_t\right]$.

Applying Theorem 1 element-wise, we have $\underline{\mathcal{L}}_t \subseteq \Phi\left(\overline{\mathcal{L}}_t, z\right) \subseteq \mathcal{L}^\star(z) \subseteq \Phi\left(\underline{\mathcal{L}}_t, z\right) \subseteq \overline{\mathcal{L}}_t$. Thus, it is sufficient to show that $\Phi^g\left(\overline{\mathcal{L}}_t, z\right) = \Phi\left(\overline{\mathcal{L}}_t, z\right)$ and $\Phi^g\left(\underline{\mathcal{L}}_t, z\right) = \Phi\left(\underline{\mathcal{L}}_t, z\right)$ for every $z \in \mathcal{Z}_t$, since $\mathcal{Z}_t$ was an arbitrary interval of $\mathcal{T}$. Choose an arbitrary type $z \in \mathcal{Z}_t$ and let $\ell \in \Phi^g\left(\overline{\mathcal{L}}_t, z\right)$ be an arbitrary element. Then, $z_\ell^g\left(\overline{\mathcal{L}}_t\right) < z$ and $0 = D_\ell f\left(\overline{\mathcal{L}}_t, z_\ell^g\left(\overline{\mathcal{L}}_t\right)\right)$ together imply $0 \leq D_\ell\left(\overline{\mathcal{L}}_t, z\right)$ by SCD-T. Thus, $\ell \in \Phi^g\left(\overline{\mathcal{L}}_t, z\right)$. Since ℓ was an arbitrarily chosen element of $\Phi^g(\underline{\mathcal{L}}_t, z)$, it follows that $\Phi^g(\underline{\mathcal{L}}_t, z) \subseteq \Phi(\underline{\mathcal{L}}_t, z)$ for all $z \in \mathcal{Z}_t$. Similarly, if $\ell \in \Phi(\underline{\mathcal{L}}_t, z)$ is an arbitrary element, then $0 \leq D_\ell f(\underline{\mathcal{L}}_t, z)$ implies $z \geq z_\ell^g(\underline{\mathcal{L}}_t)$ by SCD-T. Thus, $\ell \in \Phi^g(\underline{\mathcal{L}}_t, z)$ so $\Phi(\underline{\mathcal{L}}_t, z) \subseteq \Phi^g(\underline{\mathcal{L}}_t, z)$. This argument establishes that $\Phi^g\left(\overline{\mathcal{L}}_t, z\right) = \Phi\left(\overline{\mathcal{L}}_t, z\right)$ while a similar argument establishes $\Phi^g(\underline{\mathcal{L}}_t, z) = \Phi(\underline{\mathcal{L}}_t, z)$. The proof is complete for the case with SCD-C from above.

The argument in the case of SCD-C from below follows the same logic. □

D.3. Other Results

Assumption 1 (Continuity of objective in type)**.** *For all strategies $\mathcal{L}$, the function $f(\mathcal{L}, \cdot)$ is continuous in type z.*

Assumption 2 (Finite crossing of objective)**.** *For any pair of strategies $(\mathcal{L}_1, \mathcal{L}_2)$, there is a finite number of types z so that $f(\mathcal{L}_1, z) - f(\mathcal{L}_2, z) = 0$.*

Theorem 4 (Continuous policy function almost everywhere)**.** *Suppose Assumptions 1–2 hold and L is finite. Then, the policy correspondence is a function almost everywhere, and it is continuous both to the left and to the right. In particular,*

1. *Assumption 2 and finite L implies that the set of types where $\mathcal{L}^*(\cdot)$ is not unique is finite;*
2. *Assumption 1 implies that, if $\mathcal{L}^\star(z)$ is unique, then there exists $\delta > 0$ so that for all $z' \in [z - \delta, x + \delta]$, the optimal strategy is also $\mathcal{L}^\star(z)$; and*
3. *Assumptions 1–2 and finite L imply that, if the optimal strategy for z is not unique,*
   a) *there exists a unique strategy $\mathcal{L}_+$ where there exists a $\delta_+ > 0$ so that $\mathcal{L}_+$ is optimal for all $z' \in [z, z + \delta]$; and*
   b) *there exists a unique strategy $\mathcal{L}_-$ where there exists a $\delta_- > 0$ so that $\mathcal{L}_-$ is optimal for all $z' \in [z - \delta, z]$.*

Proof. We show each statement in turn.

1. If there are zero or one types wheres where $\mathcal{L}^\star(z)$ is not unique, then the theorem holds trivially. Suppose there is more than one type where the optimal strategy is not unique. Let

$$n = \max_{(\mathcal{L}_1, \mathcal{L}_2)} |\{z \mid f(\mathcal{L}_1, z) = f(\mathcal{L}_2, z)\}|$$

   where note the maximum exists because there are a finite number of pairs $(\mathcal{L}_1, \mathcal{L}_2)$. Then, there cannot be more than $n \times |\mathcal{P}(\mathcal{P}(L))|$ additional types where the optimal strategy is not unique.

2. Define

$$g\left(z'\right)=\min_{\mathcal{L}\neq\mathcal{L}^{\star}(z)}\left\{f\left(\mathcal{L}^{\star}\left(z\right),z'\right)-f\left(\mathcal{L},z'\right)\right\}$$

and observe that $g\left(z\right)>0$ since $\mathcal{L}^{\star}\left(z\right)$ is unique. Moreover, g is continuous since differences of continuous functions are continuous, and so is a minimum over a set of continuous function. Then, there is a $\tilde{\delta}>0$ so that, for all $z'\in\left(z-\tilde{\delta},z+\tilde{\delta}\right)$,

$$\begin{aligned}&\left|g\left(z'\right)-g\left(z\right)\right|<\frac{1}{2}g\left(z\right)\\&\qquad\Rightarrow g\left(z'\right)>g\left(z\right)-\frac{1}{2}g\left(z\right)=\frac{1}{2}g\left(z\right)>0\,.\end{aligned}$$

Then, for all $z'\in\left[z-\frac{1}{2}\tilde{\delta},z+\frac{1}{2}\tilde{\delta}\right]$,

$$0<f\left(\mathcal{L}^{\star}\left(z\right),z'\right)-f\left(\mathcal{L},z'\right)\qquad\qquad\forall\mathcal{L}\neq\mathcal{L}^{\star}\left(z\right)$$

so setting $\delta=\frac{1}{2}\tilde{\delta}$ is sufficient.

3. We only prove the first statement; the proof of the second is identical in spirit. Use induction on the number of optimal strategies. Suppose there are exactly two optimal strategies $\mathcal{L}_1$ and $\mathcal{L}_2$ for the type z. For $i\in\{1,2\}$, let

$$g_i\left(z'\right)=\min_{\mathcal{L}\notin\{\mathcal{L}_1,\mathcal{L}_2\}}\left\{f\left(\mathcal{L}_1,z'\right)-f\left(\mathcal{L},z'\right)\right\}$$

and observe by the same arguments as above that $\tilde{g}\equiv g_2-g_1$ is continuous. First, suppose $\tilde{g}\left(z'\right)\neq 0$ for all $z'>z$. By the continuous value theorem, it must either be that $\tilde{g}\left(z'\right)>0$ or $\tilde{g}\left(z'\right)<0$. Assign

$$\mathcal{L}_{+}=\begin{cases}\mathcal{L}_2 & \text{if } \tilde{g}\left(z'\right)>0 \text{ for } z'>z\\ \mathcal{L}_1 & \text{if } \tilde{g}\left(z'\right)<0 \text{ for } z'>z\end{cases}$$

Note that $\mathcal{L}_{+}$ is optimal for all $z'>z$. Then, the statement holds for any positive value of δ. Now, suppose $\tilde{g}\left(z'\right)=0$ for some $z'>z$. Let

$$\underline{z}\equiv\max\left\{z'\mid z'>z,\tilde{g}\left(z'\right)=0\right\}>z$$

which exists since the set is finite by Assumption 2. Observe that for any $z'\in\left(z,\frac{1}{2}z+\frac{1}{2}\underline{z}\right]$, we have that $\tilde{g}\left(z'\right)\neq 0$. In the same argument as above, $\tilde{g}\left(z'\right)>0$ for the entire interval or $\tilde{g}(z') < 0$. Thus, assigning $\delta = \frac{1}{2}\underline{z} - \frac{1}{2}z$ is sufficient. We have now established the base case. Now suppose the statement holds if there are k optimal strategies at z. Consider the case where there are $k+1$ optimal strategies, enumerated $\mathcal{L}_1, \ldots, \mathcal{L}_{k+1}$, and the modified objective function

$$\tilde{f}(\mathcal{L}, z) = \begin{cases} f(\mathcal{L}, z) & \text{if } \mathcal{L} \neq \mathcal{L}_{k+1} \\ f(\mathcal{L}_1, z) & \text{if } \mathcal{L} = \mathcal{L}_{k+1} \end{cases}$$

noting that, for $z' = z$, the first k strategies are optimal for the modified problem. Then, the statement holds for the modified problem; let $\mathcal{L}_+ \in \{\mathcal{L}_i \mid i = 1, \ldots, k\}$ be the strategy for which the statement holds in the modified problem. Repeat the proof for the base case assigning $\mathcal{L}_1 = \mathcal{L}_+$ and $\mathcal{L}_2 = \mathcal{L}_{k+1}$.

□

Lemma 1. *Suppose Assumptions 1 and 3 hold. Then, for any interval $[z, z')$ of the type space, let*

$$\mathcal{L} \in \arg\max f(\mathcal{L}, z) \qquad , \qquad \mathcal{L}' \in \arg\max f\left(\mathcal{L}, z'\right)$$

be arbitrary elements of the argmaxes. If $\mathcal{L} = \mathcal{L}'$, then $\mathcal{L}^\star(\cdot) = \mathcal{L}$ on the interval $[z, z')$.

Proof. We proceed by contradiction. Let $\tilde{\mathcal{L}}$ be an argmax (possibly but not necessarily unique) for both types z and z'. Suppose there exists a type $m \in [z, z')$ for which $\mathcal{L}^\star(m) \neq \tilde{\mathcal{L}}$. If $\tilde{\mathcal{L}}$ is not optimal for the type m, then define $g(z) \equiv f(\mathcal{L}^\star(m), z) - f\left(\tilde{\mathcal{L}}, z\right)$ and note by assumption $g(m) > 0$. In addition, $g(z) \leq 0$ and $g(z') \leq 0$ by construction. Then, by the continuous value theorem, g crosses 0 at least once on the interval $[z, m)$ and again on $(m, z']$, violating Assumption 3. Then, $\tilde{\mathcal{L}}$ must be optimal for m. □

E. Policy Function Refinement

In this section, we provide details about refining the policy function after the generalized squeezing step has converged. We describe a generalization of the branching procedure, then an alternative refinement which requires additional structure on the objective function.

E.1. The Generalized Branching Procedure

We can extend the logic of the branching step defined above to the context of heterogeneous agent types.

Consider the bounding set functions $\left[\underline{\mathcal{L}}(\cdot), \overline{\mathcal{L}}(\cdot)\right]$ that result after applying the generalized squeezing procedure. Similarly to the branching step, for each z, we select one undetermined item in $\overline{\mathcal{L}}(z) \setminus \underline{\mathcal{L}}(z)$ and divide the problem into two subproblems: one which includes it for type z and one which excludes it. Then, generalized squeezing can be applied to each subproblem to determine conditionally optimal behavior. As above, the generalized branching step can be applied recursively until $\underline{\mathcal{L}}(\cdot) = \overline{\mathcal{L}}(\cdot)$ on each branch.

We now formally define the generalized branching step which uses the generalized squeezing step from Definition 8.

Definition (Generalized branching step)**.** Given bounding set functions $\left[\underline{\mathcal{L}}(\cdot), \overline{\mathcal{L}}(\cdot)\right]$, define the function $\ell(\cdot)$

$$\ell(z) = \begin{cases} \varnothing & \text{if } \underline{\mathcal{L}}(z) = \overline{\mathcal{L}}(z) \\ \{\ell\} & \text{for any } \ell \in \overline{\mathcal{L}}(z) \setminus \underline{\mathcal{L}}(z) \end{cases}$$

for each z. The mapping B is given by

$$B_\ell\left(\left[\underline{\mathcal{L}}(\cdot), \overline{\mathcal{L}}(\cdot)\right]\right) = \left\{S^K\left(\left[\underline{\mathcal{L}}(\cdot) \cup \ell(\cdot), \overline{\mathcal{L}}(\cdot)\right]\right), S^K\left(\left[\underline{\mathcal{L}}(\cdot), \overline{\mathcal{L}}(\cdot) \setminus \ell(\cdot)\right]\right)\right\} .$$

For initial bounding set functions $\left[\underline{\mathcal{L}}(\cdot), \overline{\mathcal{L}}(\cdot)\right]$, we denote the operator of applying the branching step until global convergence by $\Lambda\left(\left[\underline{\mathcal{L}}(\cdot), \overline{\mathcal{L}}(\cdot)\right]\right)$. Global convergence of the branching step occurs when the stopping condition $\underline{\mathcal{L}}(\cdot) = \overline{\mathcal{L}}(\cdot)$ is met on each branch. The globally converged result is a collection of branch-specific policy functions.

If the initial bounding set functions are outcomes from the generalized squeezing procedure, then the generalized squeezing procedure can be simplified. In particular, recall that the bounding set functions $\left[\underline{\mathcal{L}}(\cdot), \overline{\mathcal{L}}(\cdot)\right]$ induce a finite partitioning on the type space, where each interval $\mathcal{Z}_t$ is associated with a constant bounding pair $\left[\underline{\mathcal{L}}_t, \overline{\mathcal{L}}_t\right]$ shared by all types in the interval. For any interval where $\underline{\mathcal{L}}_t = \overline{\mathcal{L}}_t$, the optimal decision set has been found and no branching is required. The problem is instead to compute the policy function on intervals where $\underline{\mathcal{L}}_t \subset \overline{\mathcal{L}}_t$. We formalize this notion with the definition below.

Definition. Given a function $\mathcal{L}^\star : Z \to \mathcal{P}(L)$ and a restriction of the original domain $\mathcal{Z} \subset Z$, let $\mathcal{L}^\star|_{\mathcal{Z}}: \mathcal{Z} \to \mathcal{P}(L)$ be the function restricted to the subdomain $\mathcal{Z}$.

The generalized squeezing procedure identifies $\mathcal{L}^\star|_{\mathcal{Z}_t}$ for any interval $\mathcal{Z}_t$ where $\underline{\mathcal{L}}_t = \overline{\mathcal{L}}_t$. The generalized branching procedure thus need only be applied to a interval $\mathcal{Z}_t$ where $\underline{\mathcal{L}}_t \subset \overline{\mathcal{L}}_t$, which identifies $\mathcal{L}^\star|_{\mathcal{Z}_t}$ without needing to perform unnecessary computation on intervals for which the policy function has already been identified.

Figure 9 shows an example of applying the generalized branching step for an objective function satisfying SCD-C and SCD-T. In this example, the generalized squeezing procedure

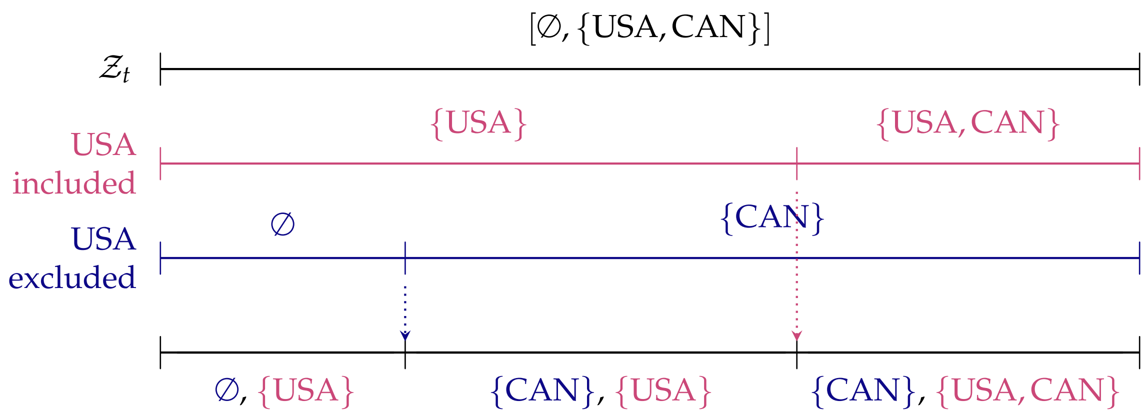


FIGURE 9: AN EXAMPLE OUTCOME OF THE GENERALIZED BRANCHING PROCEDURE

The figure illustrates an possible outcome of the generalized branching procedure, in an example where $\underline{\mathcal{L}}(\cdot) = \emptyset$ and $\overline{\mathcal{L}}(\cdot) = \{\text{USA}, \text{CAN}\}$. The branching step selects $\ell(\cdot) = \{\text{USA}\}$ and creates two branches: one which presumes USA is in the optimal set and the other which presumes the opposite. Convergence on a single branch occurs when the generalized squeezing procedure returna conditionally optimal policy function. The final output of the full recursive squeezing procedure is the collection of all conditionally optimal policy functions.

has converged, but on the interval $\mathcal{Z}_t$, the bounding set functions are not equal. In particular, $\underline{\mathcal{L}}(z) = \emptyset$ and $\overline{\mathcal{L}}(z) = \{\text{USA}, \text{CAN}\}$ for all $z \in \mathcal{Z}_t$. This starting point is depicted at the top of the figure. For all other intervals $\mathcal{Z}_{t'}$, the policy function has been identified by generalized squeezing and therefore do not require branching.

The generalized branching step then forms two branches, which are depicted immediately below: the blue branch presumes that USA is in the optimal set along the interval while the pink branch presumes that it is not.

Applying the generalized squeezing procedure to both branches results in a conditional policy function for each branch. Finally, at the bottom of the figure, the two conditional policy functions are merged to deliver the solution candidates for each sub-interval: in the left sub-interval, the conditional policy function from the branch that excludes USA yields the solution candidate $\emptyset$ while the conditional policy function from the branch that includes USA yields the solution candidate $\{\text{USA}\}$. A similar logic holds for the other two sub-intervals. Then, the unconditional policy function selects among the solution candidates, for each $z \in \mathcal{Z}_t$, the decision set which delivers the highest value for the objective function.

If it were the case that one branch contains a sub-interval in which the generalized squeezing step has not identified the policy function, then the generalized branching procedure only need be applied recursively to that particular sub-interval.

To summarize the branching procedure, let $\left[\underline{\mathcal{L}}(\cdot), \overline{\mathcal{L}}(\cdot)\right]$ be valid bounding set functions.

Then, whenever f exhibits both SCD-C and SCD-T,

$$\mathcal{L}^{\star}(z) = \arg \max_{\left\{\mathcal{L}(z) | \mathcal{L}(\cdot) \in \Lambda\left(\left[\underline{\mathcal{L}}(\cdot), \overline{\mathcal{L}}(\cdot)\right]\right)\right\}} f(\mathcal{L}, z) \ .$$

For each type z, the properties of the branching step from the previous section hold.

E.2. Iterative Cutoff Search

In this section, we describe a method of choosing

$$\mathcal{L}(t) = \arg \max_{\mathcal{L} \in S} f(\mathcal{L}, t)$$

on the interval $t \in [z, z')$. To do so, we introduce a final Assumption on the objective function.

Assumption 3 (Single crossing of objective)**.** *For any pair of strategies $(\mathcal{L}_1, \mathcal{L}_2)$, there is at most one type z so that $f(\mathcal{L}_1, z) = f(\mathcal{L}_2, z)$.*

The iterative cutoff search described in this section requires the policy function to satisfy Assumptions 1 and 3. It can thus requires structure additional to SCD-T from the main text.[18]

Definition 9 (Iterative cutoff search)**.** *Consider an interval $[z, z')$ and choices S. Set $\ell^{(1)} = z$, $r^{(1)} = z'$, $\mathcal{L}_{\ell}^{(1)} = \mathcal{L}^{\star}(z)$, and $\mathcal{L}_{r}^{(1)} = \mathcal{L}^{\star}(z')$. Iterate as follows.*

1. *Identify the type $m^{(n)}$ so that*

$$f\left(\mathcal{L}_{\ell}^{(n)}, m^{(n)}\right) = f\left(\mathcal{L}_{r}^{(n)}, m^{(n)}\right)$$

e.g. from single-agent crossing, as well as an element $\mathcal{L}_{m}^{(n)} \in \arg\max_S f\left(\mathcal{L}, m^{(n)}\right)$.

   a) *If $\mathcal{L}_{m}^{(n)}$ coincides with neither $\mathcal{L}_{\ell}^{(n)}$ nor $\mathcal{L}_{r}^{(n)}$, set*

$$\begin{aligned} \ell^{(n+1)} &= \ell^{(n)} \quad , \quad & \mathcal{L}_{\ell}^{(n+1)} &= \mathcal{L}_{\ell}^{(n)} \\ r^{(n+1)} &= m^{(n)} \quad , \quad & \mathcal{L}_{r}^{(n+1)} &= \mathcal{L}_{m}^{(n)} \end{aligned}$$

   and return to the first step.

   b) *Otherwise, set*

$$\mathcal{L}^{\star}(z) = \begin{cases} \mathcal{L}_{\ell}^{(n)} & \textit{for } z \in \left[\ell^{(n)}, m^{(n)}\right) \\ \mathcal{L}_{r}^{(n)} & \textit{for } z \in \left[m^{(n)}, r^{(n)}\right) \end{cases} .$$

[18] Note that, as long as Assumption 3 holds, then the objective function also satisfies SCD-T.

2. *If $r^{(n)} = z'$, the interval has been resolved. Otherwise, set*

$$\ell^{(n+1)} = r^{(n)} \quad , \quad \mathcal{L}_\ell^{(n+1)} = \mathcal{L}_r^{(n)}$$
$$r^{(n+1)} = z' \quad , \quad \mathcal{L}_r^{(k+1)} = \mathcal{L}^\star\left(z'\right)$$

and return to the first step.

Lemma 2. *Suppose Assumption 1 holds. At all iterations n of iterative cutoff search as in Definition 9, it is the case that $z \leq \ell^{(n)} \leq r^{(n)} \leq z'$. Moreover, for each n, $m^{(n)} \in \left[\ell^{(n)}, r^{(n)}\right]$ is unique.*

Proof. We start with the first statement, using induction on n. In the first step, $z = \ell^{(1)} < r^{(1)} = z'$ trivially. Suppose the statement is true for iteration n. Consider iteration $(n+1)$. Then, either

$$\ell^{(n+1)} = r^{(n)} \neq z' \quad , \quad r^{(n+1)} = z'$$

or

$$\ell^{(n+1)} = \ell^{(n)} \quad , \quad r^{(n+1)} = m^{(n)}$$

from the previous iteration. Start with the first possibility. By inductive assumption, $z \leq r^{(n)} < r$ so $z \leq \ell^{(n+1)} < r^{(n+1)} = z'$. Now consider the second possibility. By inductive assumption, $z \leq \ell^{(n)} = \ell^{(n+1)} \leq m^{(n)} = r^{(n+1)} \leq z'$.

It must be that $\mathcal{L}_\ell^{(n)} \neq \mathcal{L}_r^{(n)}$. Then, define $g\left(z\right) \equiv f\left(\mathcal{L}_\ell^{(n)}, z\right) - f\left(\mathcal{L}_r^{(n)}, z\right)$ and note that $g\left(\ell^{(n)}\right) \geq 0$ but $g\left(r^{(n)}\right) \leq 0$, which at least one inequality strict. If $g\left(\ell^{(n)}\right) = 0$, then setting $m^{(n)} = \ell^{(n)}$ is sufficient and similarly if $g\left(r^{(n)}\right) = 0$. If both inequalities are strict, then the continuous value theorem guarantees a value $m^{(n)} \in \left(\ell^{(n)}, r^{(n)}\right)$ where $g\left(m^{(n)}\right) = 0$. It is unique by Assumption 3. □

Proposition. *If Assumptions 1 and 3 hold and S is finite, then iterative cutoff search as in Definition 9 correctly identifies $\mathcal{L}^\star\left(\cdot\right)$ on $[z, z')$.*

Proof. Note that iterative search concludes in a finite number of iterations, since S is finite and Assumption 3 guarantees at most $\binom{|S|}{2}$ cutoffs.

Use strong induction on the number of iterations it takes to conclude. Consider the case where the iteration concludes in one step. By Lemma 2, $m \in \left[\ell^{(1)}, r^{(1)}\right]$. It also must be that $\mathcal{L}_m^{(1)}$ coincides with either $\mathcal{L}_\ell^{(1)}$ or $\mathcal{L}_r^{(1)}$. Suppose it coincides with $\mathcal{L}_\ell^{(1)}$. Additionally,

$$f\left(\mathcal{L}_m^{(1)}, m^{(1)}\right) = f\left(\mathcal{L}_\ell^{(1)}, m^{(1)}\right) = f\left(\mathcal{L}_r^{(1)}, m^{(1)}\right)$$

where the last equality follows from the definition of $m^{(1)}$. By Lemma 1, the policy function is thus set correctly along the entire interval. A similar argument follows if $\mathcal{L}_m^{(1)}$ coincides instead with $\mathcal{L}_r^{(1)}$.

Suppose, if the search concludes in $1 \leq k \leq n$ iterations, then it correctly identifies the policy function. Consider the case where it concludes in $(n+1)$ iterations. The first iteration must bypass step 1b, since otherwise the search concludes. Then, it must be that

$$\ell^{(2)} = \ell^{(1)} = z \qquad\qquad , \qquad\qquad r^{(2)} = m^{(1)} .$$

Consider alternatively initiating the iteration on $\left[z, m^{(1)}\right)$ and call this search the "left search". Note that left search proceeds identically to the original iteration from iteration 2. Since the original iteration concludes in $(n+1)$ iterations, left search takes at most n steps. There is thus some $k < n$ where $r^{(k+1)} = m^{(1)}$. By strong inductive assumption, left search correctly determines the policy function on $\left[z, m^{(1)}\right)$.

Once left search concluded on the kth iteration, the iteration sets $\ell^{(k+1)} = m^{(1)}$ and $r^{(k+1)} = z'$. Iterations $(k+1)$ to $(n+1)$ proceed as if the iteration had been initiated with $\left[m^{(1)}, z'\right)$. Similarly by the strong inductive assumption, iterative cutoff search correctly determines the policy function on this interval. □

Given a objective function that satisfies Assumptions 1 and 3, iterative cutoff search can thus be used. Setting $S = \mathcal{P}(L)$, it can be directly applied to find the policy function on any finite interval $[z, z')$ for which the policy function at the endpoints is known. Alternatively, it can be used after generalized squeezing has converged to refine the policy function on each sub-interval $\mathcal{Z}_t$ for which the bounding pair is loose. Finally, it can be used after generalized squeezing and branching to identify the policy function on each sub-interval where $\Lambda\left(\left[\underline{\mathcal{L}}(\cdot), \overline{\mathcal{L}}(\cdot)\right]\right)$ features multiple conditionally-optimal decisions.

F. SCD-C and Cross-location Employment Elasticity

In Section 3, we provide the parameter restriction which determines the direction of complementarities in the model. In particular, if the elasticity of substitution among locations in the firm's cost function, θ, exceeds the elasticity of demand, $\sigma - 1$, then the complementarities are negative; otherwise they are positive.

We give an employment elasticity interpretation of this restriction. Consider the partial equilibrium response of the firm's total employment at location $\ell' \in \mathcal{L}$ to a small change in the wage w_ℓ in location $\ell \in \mathcal{L}$, holding fixed the firm's decision set $\mathcal{L}$ and all other aggregates.

This employment elasticity is

$$\left.\frac{\partial \ln (\text{emp. in } \ell')}{\partial \ln w_\ell}\right|_{\mathcal{L}} = [\theta - (\sigma - 1)] \sum_n \left[\frac{\xi_{i\ell' n}^{-\theta} y_{in} (\mathcal{L}, z)}{\sum_{n'} \xi_{i\ell' n'}^{-\theta} y_{in'} (\mathcal{L}, z)}\right] \left[\frac{\xi_{i\ell n}^{-\theta}}{\sum_{k \in \mathcal{L}} \xi_{ikn}^{-\theta}}\right] - (\theta + 1) \mathbb{1} [\ell = \ell']$$

where $y_{in} (\mathcal{L}, z)$ are the firm's total sales in market n.

The own-elasticity, when $\ell' = \ell$, is always negative: as the wage increases, the firm adjusts downwards its employment in that location, all else equal. However, the sign of the cross-location employment elasticity, when $\ell' \neq \ell$, precisely depends on the size of θ relative to $(\sigma - 1)$. If the cross-elasticity is positive, so that the firm increases employment at all other locations when the wage at a given location increases, then there are negative complementarities among locations. This case corresponds to $\theta > \sigma - 1$, the condition for the firm's problem to satisfy SCD-C from above. On the other hand, if the firm instead decreases employment at all other locations, then there are positive complementarities among the locations. This case corresponds to $\theta < \sigma - 1$, when the firm's problem satisfies SCD-C from below. These elasticities are the related to those estimated in Muendler and Becker (2010), but not directly comparable since our elasticities depend on the firm's particular location set $\mathcal{L}$ and the location-market shares of each (ℓ, n) pair.

G. Generalized Theoretical Framework

In this section, we relax the assumptions on the production structure and the demand system in our model in Section 3. The generalized cost and demand functions nest several prominent frameworks. We then show how to establish both single-crossing differences conditions in this more general setup.

G.1. General Cost Function

Consider a firm of productivity $z \in \mathbb{R}^+$ headquartered in country i with a production location set $\mathcal{L}$ and the unit cost $c_{in} (\mathcal{L}, z)$ of delivering its final good to a destination market n. In Section 3, equation (1), we presented a particular formulation for $c_{in} (\mathcal{L}, z)$, which we microfound at the beginning of this section. Here, we relax the assumption on $c_{in} (\mathcal{L}, z)$ while remaining agnostic on its microfoundation.

Assumption 4 (Generalized marginal cost function)**.** *The marginal cost function of a firm headquartered in country i with productivity z in destination n can be written as the composition $c_{in} (\mathcal{L}) = g (\mathbf{\Theta}_{in} (\mathcal{L}), z)$ of the vector-valued production index function $\mathbf{\Theta}_{in} : \mathcal{P}(\mathcal{L}) \to \mathbb{R}^K$ and the outer cost function $g : \mathbb{R}^K \times \mathbb{R}^+ \to \mathbb{R}^+$ where:*

1. *each dimension of the production index function features no interdependencies among elements of $\mathcal{L}$ and is increasing in $\mathcal{L}$, i.e. for all $\ell \in L$, $\mathcal{L} \subseteq L$, and $k \leq K$,*

$$\xi_{ki\ell n} \equiv \Theta_{kin}\left(\mathcal{L} \cup \{\ell\}\right) - \Theta_{kin}\left(\mathcal{L} \setminus \{\ell\}\right) \geq 0$$

   is independent of $\mathcal{L}$; and

2. *the outer cost function g is monotonically decreasing in each dimension of production potential and in firm productivity (if it is increasing, redefine $\tilde{z} \equiv -z$), i.e. for all $\mathcal{L} \subseteq L$, $k \leq K$, and $z \in \mathbb{R}^+$*

$$\frac{\partial g}{\partial \Theta_{kin}} \leq 0 \qquad , \qquad \frac{\partial g}{\partial z} \leq 0\,.$$

The central object in Assumption 4 is the production "index" function $\boldsymbol{\Theta}_{in}$ that measure the overall potential of the production location set $\mathcal{L}$ along K dimensions. These dimensions could represent different technological techniques of production, industries, or other latent variables. Many papers in the multinational literature refer to $\boldsymbol{\Theta}_{in}\left(\mathcal{L}\right)$ as the "production potential" or "sourcing potential" associated with a given location set (see, e.g., Antras, Fort, and Tintelnot 2017). Importantly, for each dimension k, the marginal contribution of each location to the index is independent of the marginal contribution of other locations.

We now present the assumption on fixed costs.

Assumption 5 (General fixed cost function)**.** *The total fixed cost of establishing a production location set $\mathcal{L}$ for a firm headquartered in location i, $F_i\left(\mathcal{L}\right)$, is given by:*

$$F_i\left(\mathcal{L}\right) = \sum_{\ell \in \mathcal{L}} F_{i\ell}.$$

Assumption 5 asserts that there is an independent fixed cost F_{ij} for establishing each production location. In the framework of Section 3, these fixed costs are $F_{i\ell} = w_\ell f_{i\ell}$.

Special cases We discuss existing frameworks which satisfy the structure imposed in Assumption 4. Consider first the case where $K = 1$, so the production index $\Theta_{in}\left(\mathcal{L}\right)$ is a scalar. The CES marginal cost function $c_{in}\left(\mathcal{L}, z\right)$ in Section 3, as well as in Antras, Fort, and Tintelnot (2017) and Tintelnot (2017), follow this structure with

$$\Theta_{in}\left(\mathcal{L}\right) = \sum_{\ell \in \mathcal{L}} \xi_{i\ell n} \qquad , \qquad g\left(\Theta, z\right) = \frac{\Gamma}{z}\Theta^{-\frac{1}{\theta}}$$

where $\xi_{i\ell n}$ is a combination of fundamentals and aggregates, while Γ is a constant of integration. Assumption 4 is also satisfied in models in which the location-input-specific productivity shocks are distributed according to a multivariate Pareto as in Arkolakis, Ramondo, et al. (2018), or a Fréchet distribution with a uniform correlation across draws. In these cases, the index function remains the same while the outer cost function is

$$g\left(\Theta, z\right) = \frac{\Gamma}{z}\Theta^{-\frac{1-\rho}{\theta}}$$

where ρ is the correlation among draws and θ is the distribution's shape parameter.

Lind and Ramondo (2023) present a cost function that features K nests. This cost function satisfies the multidimensional case with $K > 1$. In this case,

$$\Theta_{kin}\left(\mathcal{L}\right) = \sum_{\mathcal{L}} \xi_{ki\ell n} \qquad , \qquad g\left(\boldsymbol{\Theta}, z\right) = \frac{\Gamma}{z}\left[\sum_{k} \Theta_{kin}\left(\mathcal{L}\right)^{1-\rho_k}\right]^{-\frac{1}{\theta}}$$

where now the location set $\mathcal{L}$ maps to a different potential for each technique k and ρ_k is the substitutability across locations within the nest k. As an example, suppose there is a standard production technique $k = 1$ and a skill-intensive production technique $k = 2$. In this case, $K = 2$, and the nests represent production techniques. Then, $\Theta_{1in}\left(\mathcal{L}\right)$ represents overall potential of the location set $\mathcal{L}$ when the firm applies the standard production technique, while $\Theta_{2in}\left(\mathcal{L}\right)$ represents the potential when applying the skill-intensive technique. These potentials differ since the low-skill and high-skill wages could differ in each location, so the potential of a particular location set depends on which technique the firm uses. In the same way, locations substitute for each other within each production technique, captured by ρ_k, but not directly across techniques. Lind and Ramondo (2023) microfound this cost function using a nested multivariate Fréchet distribution with correlations ρ_k, and show that is sufficiently flexible to approximate any general correlation structure up to arbitrarily close precision. Note that this formulation nests the previous special cases when $K = 1$.

G.2. General Demand Function

Consider a set of destination markets N, each of which feature consumers with residual demand function $q_n\left(p_n\right)$. We then impose the following structure on the firm's variable profits.

Assumption 6 (Generalized variable profit function)**.** *Given pricing choices $\left\{p_n\right\}_n$ in each market n, the variable profits of a firm headquartered in location i take the form*

$$v\left(\boldsymbol{c}_i\right) = \sum_{n \in N} \left[q_n\left(p_n\right) p_n - q_n\left(p_n\right) c_{in}\right],$$

where $\mathbf{c}_i = \left[c_{in}\right]_n$ is the vector of unit costs of producing and delivering a good to the destination markets

n.

A key feature of this profit function is that the destination markets are independent and that there are no strategic interactions among firms. In particular, the unit cost c_{in} of serving a market n does not affect the variable profits earned in destination market $n' \neq n$. Similarly, the price p_n set by the firm in market n does not affect the variable profits in a different destination market. This formulation does not require demand to be homothetic, nor does it place any particular restrictions on the elasticity of demand.

Following standard firm maximization, the firm sets a different price in each market according to the rule

$$p_n^* \left(c_{in}\right) = \frac{\varepsilon_{q_n} \left(p_n^* \left(c_{in}\right)\right)}{\varepsilon_{q_n} \left(p_n^* \left(c_{in}\right)\right) - 1} c_{in},$$

where $\varepsilon_{q_n}\left(p\right)$ is the price elasticity of the demand function q_n at the price p. Incorporating the optimal pricing rule, we define the variable profits in market n earned at the optimal price

$$v_n^* \left(c_{in}\right) \equiv q_n \left(p_n^* \left(c_{in}\right)\right) p_n^* \left(c_{in}\right) - q_n \left(p_n^* \left(c_{in}\right)\right) c_{in}$$

as a function of marginal cost c_{in}.

Special cases Our framework from Section 3 posits the constant elasticity (CES) demand system, which easily satisfies Assumption 6 with $q_n\left(p_n\right) = \frac{X_n}{P_n}\left(\frac{p_n}{P_n}\right)^{-\sigma_n}$ where X_n and P_n are market aggregates. The optimal pricing rule is $p_n^\star\left(c_n\right) = \frac{\sigma_n}{\sigma_n - 1} c_n$, which implies constant markups over marginal costs.

Assumption 6 is sufficiently general to allow variable elasticity of demand and thus variable markups. As an illustrative example, we discuss the Pollak (1971) demand system which is also satisfies Assumption 6 and has become popular in the literature studying variable markups.[19] The demand function is characterized by

$$q_n\left(p_n\right) = \left(\frac{p_n}{P_n^\star}\right)^{-\sigma_n} + \gamma \qquad , \qquad p_n^\star\left(c_n\right) = \frac{\sigma_n}{\left(\sigma_n - 1\right) + \left(\frac{p_n^\star\left(c_n\right)}{P_n^\star}\right)^{\sigma_n}} c_n \tag{12}$$

where $\gamma < 0$ and $P_n^\star$ is the market aggregate choke price. The markup is decreasing in the firm's marginal cost.

The demand in equation (12) features several appealing features for the study of variable markups. First, there is a choke price, which implies that entry into each destination market n is guaranteed only for the firms with low enough marginal costs $c_n \leq P_n^\star$. A firm does

[19] See, for example, Simonovska (2015), Klenow and Willis (2016), Arkolakis, Costinot, Donaldson, et al. (2019), and Behrens et al. (2020).

not necessarily serve all countries, but self-selects into export markets consistent with the data (see also Arkolakis, Costinot, Donaldson, et al. 2019). Second, the elasticity of demand asymptotically approaches constant, which allows the model to fit the Pareto tails of firm size distribution, a key feature of the data (see Amiti, Itskhoki, and Konings 2019; Arkolakis 2016). Finally, under very general conditions, it implies that markups increase with firm size, a salient finding of recent investigations on the relationship of firm size and firm markups (see De Loecker et al. 2016).

G.3. Sufficient Conditions for SCD-C

Given Assumptions 4–6, the firm's variable profits across all markets n net of fixed costs is

$$\pi_i(\mathcal{L}, z) = \sum_n v_n^*(c_{in}(\mathcal{L}, z)) - F_i(\mathcal{L})$$

and thus its CDCP is to maximize this function with respect to the decision set $\mathcal{L}$.

We now derive a sufficient condition for SCD-C. To begin, we can write the marginal value of location j as defined in Definition 1 as follows:

$$\begin{aligned} D_\ell \pi_i(\mathcal{L}, z) &= \sum_n \left[v_n^*(c_{in}(\mathcal{L} \cup \{\ell\}, z)) - v_n^*(c_{in}(\mathcal{L} \setminus \{\ell\}, z))\right] - F_{i\ell} \\ &= \sum_n \int_0^1 \boldsymbol{\xi}_{in}(\ell)' \left[\nabla_{\boldsymbol{\Theta}} v_n^*(\boldsymbol{\Theta}(\mathcal{L}_1 \setminus \{\ell\}) + t\boldsymbol{\xi}_{in}(\ell), z)\right] \mathrm{d}t - F_{ij} \end{aligned}$$

where $\boldsymbol{\xi}_{in}(\ell)$ is the $K \times 1$ vector with kth element ξ_{kijn} which represents the marginal contributions of location ℓ to the production index of each technique k. The second line follows from the gradient theorem. Overall, the marginal value of a location ℓ represents the gain in variable profits from increasing the each dimension of the index function $\boldsymbol{\Theta}_{in}$, offset by the additional fixed costs incurred.

The SCD-C condition requires that marginal value only cross zero once. It is sufficient to show the marginal value is monotonic, i.e. given any $\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq L$, the marginal value of any given item ℓ is bigger (smaller) at $\mathcal{L}_2$ than for $\mathcal{L}_1$ for SCD-C from below (above). Comparing this marginal value across two decision sets $\mathcal{L}_1$ and $\mathcal{L}_2$ for any pair $\mathcal{L}_1 \subseteq \mathcal{L}_2$,

$$\begin{aligned} D_\ell \pi_i(\mathcal{L}_2) - D_\ell \pi_i(\mathcal{L}_1) = \sum_n \int_0^1 \int_0^1 \boldsymbol{\xi}_{in}(\ell)' H_{\boldsymbol{\Theta}} v_n^* (\boldsymbol{\Theta}(\mathcal{L}_1 \setminus \{\ell\}) + t\boldsymbol{\xi}_{in}(\ell) \\ + u(\boldsymbol{\Theta}(\mathcal{L}_2 \setminus \{\ell\}) - \boldsymbol{\Theta}(\mathcal{L}_1 \setminus \{\ell\})); z) \boldsymbol{\xi}_{in}(\ell) \, \mathrm{d}u \mathrm{d}t \end{aligned}$$

where H is the Hessian operator. We assume the second derivative of the profit function exists and use the fact that, as a direct consequence of the index function being a sum of marginal effects, the value of the index evaluated at $\mathcal{L}_2$ exceeds its value at $\mathcal{L}_1$ for all k.

Then, if the Hessian of v_n^* is positive semidefinite for all n, the difference is guaranteed to be positive and the firm's problem exhibits monotone complements which sufficient for SCD-C from below. On the other hand, if the Hessians for all n is negative semidefinite, the difference is guaranteed to be negative and the firm's problem exhibits monotone substitutes which sufficient for SCD-C from above. Translating this condition to restrictions on the cost and demand functions, the (k, k')th element of the Hessian $H_{\mathbf{\Theta}} v_n^*$ is as follows.

$$\frac{\partial^2 v_n^*}{\partial \Theta_{kin} \partial \Theta_{k'in}} = \underbrace{\frac{\partial v_n^*(c)}{\partial c}}_{\equiv v_n^{*\prime}} \underbrace{\frac{\partial g(\mathbf{\Theta}, z)}{\partial \Theta_{kin}}}_{\equiv g_k'} \left(-\frac{\partial \ln g(\mathbf{\Theta}, z)}{\partial \Theta_{k'in}}\right) \left[\varepsilon_{v_n^{*\prime}} - \frac{-\frac{\partial \ln g_k'(\mathbf{\Theta},z)}{\partial \ln \Theta_{k'in}}}{-\frac{\partial \ln g(\mathbf{\Theta},z)}{\partial \ln \Theta_{k'in}}}\right]$$

$$\varepsilon_{\pi_n^{\star\prime}} \equiv \frac{\frac{\partial^2 v_n^\star(c)}{\partial c^2}}{\frac{\partial v_n^\star(c)}{\partial c}} = \underbrace{\varepsilon_{q_n}(p)}_{\text{demand elasticity}} \underbrace{\frac{\mathrm{d} \ln p_n^*}{\mathrm{d} \ln c}}_{\text{passthrough}}$$

The sign of this element is entirely determined by the term in the square brackets, since the other terms are positive by assumption. We summarize the result below.

Proposition (Sufficient condition for SCD-C)**.** *Suppose the firm's problem satisfies Assumptions 4–6. Then, the following condition is sufficient for the firm problem to satisfy SCD-C from above.*

$$\underbrace{\frac{\mathrm{d} \ln q_{in}(\mathcal{L}; z)}{\mathrm{d} \ln p_{in}(\mathcal{L}; z)} \frac{\mathrm{d} \ln p_{in}(\mathcal{L}; z)}{\mathrm{d} \ln c_{in}(\mathcal{L}; z)}}_{\textit{Demand Channel}} \leq \underbrace{\frac{-\frac{\partial \ln g_k'(\mathbf{\Theta},z)}{\partial \ln \Theta_{k'in}}}{-\frac{\partial \ln g(\mathbf{\Theta},z)}{\partial \ln \Theta_{k'in}}}}_{\textit{Supply Channel}} \qquad \forall n, k, k' \tag{13}$$

Reversing the inequality yields a sufficient condition for SCD-C from above.

This condition collapses in the following special cases.

1. *In the case of CES demand, the demand channel collapses to σ.*
2. *In the case of Pollak (1971) demand, the demand channel collapses to*

$$\underbrace{\frac{\sigma_n}{1 - \left(\frac{p_n^\star}{P_n^\star}\right)^{\sigma_n}}}_{\textit{Demand Elasticity}} \underbrace{\frac{(\sigma_n - 1) + \left(\frac{p_n^\star}{P_n^\star}\right)^{\sigma_n}}{(\sigma_n - 1) + (\sigma_n + 1)\left(\frac{p_n^\star}{P_n^\star}\right)^{\sigma_n}}}_{\textit{Passthrough}}$$

which is bounded below by σ_n.

3. *In the single-dimensional cost formulation of Tintelnot (2017), Antras, Fort, and Tintelnot (2017), and Arkolakis, Ramondo, et al. (2018), the supply channel collapses to* $1 + \frac{\theta}{1-\rho}$.
4. *In the multi-dimensional formulation of Lind and Ramondo (2023), the condition becomes*

$$\begin{aligned} \frac{\mathrm{d} \ln q_{in}\left(\mathcal{L};z\right)}{\mathrm{d} \ln p_{in}\left(\mathcal{L};z\right)} \frac{\mathrm{d} \ln p_{in}\left(\mathcal{L};z\right)}{\mathrm{d} \ln c_{in}\left(\mathcal{L};z\right)} &\leq 1+\theta && \textit{for SCD-C from above} \\ &\geq 1+\theta+\frac{\rho_k}{\frac{1-\rho_k}{\theta}\frac{\Theta_{kin}^{1-\rho_k}}{\sum_j \Theta_{jin}^{1-\rho_{k'}}}} && \forall k \textit{ for SCD-C from below} \end{aligned}$$

By assumption, an additional production location always lowers the marginal cost of the firm to supply its final good to any location. The firm's problem exhibits positive complementarities when an additional locations leads to a larger profit gain the more locations the firm operates, and vice versa for negative complementarities. Equation (13) decomposes this effect into a supply-side component and a demand-side component.

The supply-side component captures how much an additional production location reduces the marginal cost of the firm, while the demand-side component captures by how much variable profits increase when the marginal cost of the firm drops. The balance of these two forces determines whether the firm's overall profit maximization problem exhibits positive or negative complementarities. The strength of the demand-side channel depends on the product of the demand and passthrough elasticity. It summarizes the elasticity of variable profits to a change in marginal cost, which is determined by how much a marginal cost change affects the price (passthrough) and in turn by how much demand responds to a marginal decrease in price (demand elasticity).

The condition in the above equation separates demand and supply side forces. The modeling assumptions in Section 3 implies that demand side forces are always inducing positive complementarities among production locations and supply side forces negative complementarities. In alternative models of multi-location production, it is possible for the cost side term to be below, if the locations are complements in cost. One microfoundation for these cost-side complementarities could be scale economies in the number of production sites, agglomeration in the density of production locations, or complementarities that may arise from location-level specialization.[20]

[20] Though it is possible for the demand-side complementarities to be negative, through either negative demand elasticity or passthrough, this scenario is less likely.

G.4. Sufficient Conditions for SCD-T

To derive a sufficient condition for SCD-T, we introduce a final Assumption on the role of productivity.

Assumption 7 (Hicks-neutral productivity)**.** *The outer function $g : \mathbb{R}^K \times \mathbb{R}^+ \to \mathbb{R}^+$ is multiplicatively separable, so that it can be written*

$$g(\mathbf{\Theta}, z) = \frac{1}{z}\tilde{g}(\mathbf{\Theta}) \ .$$

We now derive a sufficient condition for SCD-C under Assumptions 4–7.

Following Assumption 7, the marginal value of location ℓ can be rewritten

$$\begin{aligned} D_\ell \pi_i(\mathcal{L}, z) = \sum_n \int_0^1 v_n^{*\prime}\left(\frac{\tilde{g}(\mathbf{\Theta}(\mathcal{L} \setminus \{\ell\}) + \boldsymbol{\xi}_{in}(\ell)\, t)}{z}\right) \\ \times \boldsymbol{\xi}_{in}(\ell)' \nabla_{\mathbf{\Theta}} \tilde{g}(\mathbf{\Theta}(\mathcal{L} \setminus \{\ell\}) + \boldsymbol{\xi}_{in}(\ell)\, t)\frac{1}{z}\mathrm{d}t - F_{ij} \end{aligned} .$$

Comparing this marginal value for two otherwise identical agents with productivities z_1 and z_2, where $z_1 \le z_2$, it is sufficient for SCD-T to show that the marginal value of the location in ℓ is higher for the agent with the higher productivity.

$$\begin{aligned} D_\ell \pi_i(\mathcal{L}, z_2) - D_\ell \pi_i(\mathcal{L}, z_1) = \sum_n \int_{z_1}^{z_2}\int_0^1 \left[\varepsilon_{v_n^{*\prime}} - 1\right]\left\{-\frac{v_n^{*\prime}\left(\frac{\tilde{g}(\boldsymbol{\xi}_{in}(\ell) t + \mathbf{\Theta}(\mathcal{L}\setminus\{\ell\}))}{z}\right)}{z^2}\right\} \\ \times \boldsymbol{\xi}_{in}(\ell)' \nabla_{\mathbf{\Theta}} \tilde{g}(\boldsymbol{\xi}_{in}(\ell)\, t + \mathbf{\Theta}(\mathcal{L} \setminus \{\ell\}))\, \mathrm{d}t\mathrm{d}z \end{aligned}$$

Similarly to the argument for SCD-C, all terms in this expression are positive by assumption except the term in square brackets. Therefore, for this difference to be non-negative, it is sufficient for the term in the second set of square brackets to be positive. We thus derive the simple sufficiency condition for SCD-T that $\varepsilon_{v_n^{*\prime}} \ge 1$. We summarize below.

Proposition (Sufficient condition for SCD-T)**.** *Suppose the firm's problem satisfies Assumptions 4–7. Then, the following condition is sufficient for the problem to satisfy SCD-T.*

$$\frac{\mathrm{d}\ln q_{in}(\mathcal{L}; z)}{\mathrm{d}\ln p_{in}(\mathcal{L}; z)}\frac{\mathrm{d}\ln p_{in}(\mathcal{L}; z)}{\mathrm{d}\ln c_{in}(\mathcal{L}; z)} \ge 1 \qquad \forall n$$

Intuitively, the SCD-T condition requires that the variable profit increase associated with an additional production location is higher at more productive firms, akin to a cross-derivative.

This condition again separates into a demand-side effect on the left and a supply-side effect on the right. The demand-side effect is identical to the one from SCD-C, and describes the elasticity of variable profits to marginal costs. The supply-side effect captures how the reduction in marginal costs associated with an additional production location interacts with firm productivity. An additional production location is worth more at an unproductive firm compared to a productive firm, since the productive firm has high marginal costs but can shore up its low productivity by establishing more production locations. In other words, productivity and production sites are substitutes in the firm's cost function. As productivity enters the cost function multiplicatively, the elasticity of substitution between the benefit of a production location and the firm's innate productivity is simply 1.

Under the CES assumption, the condition for SCD-T collapses to $\sigma \geq 1$.

H. Computational Implementation

In this section, we describe the practical implementation of the solution method, as well as the general equilibrium framework which embeds it.

H.1. Solving CDCPs

"Eager" Squeezing Given the bounding pair $\left[\underline{\mathcal{L}}, \overline{\mathcal{L}}\right]$, the squeezing step requires computing the marginal value of each location $\ell \in L$ at both the lower and upper bounding decision sets. The computational implementation makes two modifications. First, it only computes the marginal values for locations in $\overline{\mathcal{L}} \setminus \underline{\mathcal{L}}$: locations either included in $\underline{\mathcal{L}}$ or excluded from $\overline{\mathcal{L}}$ remain included or excluded, respectively, and need not be rechecked.

Second, the squeezing step is "eager" in the sense that, once an undetermined location is known to be included or excluded, the bounding pair updates before computing the marginal value of subsequent undetermined locations. In particular, given an undetermined item $\ell \in \overline{\mathcal{L}} \setminus \underline{\mathcal{L}}$, updating occurs as follows.

$$\underline{\mathcal{L}}' = \begin{cases} \underline{\mathcal{L}} & \text{if } D_\ell f\left(\mathcal{L}^{\text{in}}\right) \leq 0 \\ \underline{\mathcal{L}} \cup \{\ell\} & \text{if } D_\ell f\left(\mathcal{L}^{\text{in}}\right) \geq 0 \end{cases} \qquad \mathcal{L}^{\text{in}} = \begin{cases} \overline{\mathcal{L}} & \text{if SCD-C above} \\ \underline{\mathcal{L}} & \text{if SCD-C below} \end{cases}$$

$$\overline{\mathcal{L}}' = \begin{cases} \underline{\mathcal{L}} \setminus \{\ell\} & \text{if } D_\ell f\left(\mathcal{L}^{\text{out}}\right) \leq 0 \\ \underline{\mathcal{L}} & \text{if } D_\ell f\left(\mathcal{L}^{\text{out}}\right) \geq 0 \end{cases}, \qquad \mathcal{L}^{\text{out}} = \begin{cases} \underline{\mathcal{L}} & \text{if SCD-C above} \\ \overline{\mathcal{L}} & \text{if SCD-C below} \end{cases}$$

The decision set $\mathcal{L}^{\text{in}}$ is the bounding set that helps determine whether the location is included; similarly, the decision $\mathcal{L}^{\text{out}}$ is the bounding set that helps determine whether the location is excluded.

Eager squeezing implies that, once ℓ is known to be included or excluded, the bounding pair tightens immediately to incorporate this information. Thus, the subsequent undetermined items are considered on the tightened bounding pair. To facilitate the eager squeezing, the computational implementation stores an auxiliary set A, which keeps track of the set of locations ℓ which have already been checked with the current bounding pair but have been neither definitely included nor excluded. The squeezing procedure thus has converged once $A = \overline{\mathcal{L}} \setminus \underline{\mathcal{L}}$: that is, the marginal value of all undetermined locations have been evaluated at the current bounding pair, and none of them can yet be definitely included or excluded.

Interval-based Generalized Squeezing and Refinement The lower and upper bounding set functions imply a partitioning $\mathcal{T}$ on the type space. The computational implementation of the policy function solution explicitly operates on this partitioning. In particular, each interval $\mathcal{Z}_t$ of the partitioning is stored separately as a tuple $\left(\mathcal{Z}_t, \underline{\mathcal{L}}_t, \overline{\mathcal{L}}_t, A_t\right)$, where $\underline{\mathcal{L}}_t$ and $\overline{\mathcal{L}}_t$ are the bounding pair specific to the interval. Then, generalized squeezing refines the partition eagerly, with the auxiliary set A_t tracking the locations in $\overline{\mathcal{L}}_t \setminus \underline{\mathcal{L}}_t$ whose marginal values have been checked at the current bounding pair but have been neither definitively included nor excluded. In particular, given a tuple $\left(\mathcal{Z}_t, \underline{\mathcal{L}}_t, \overline{\mathcal{L}}_t, A_t\right)$, the computational implementation chooses an undetermined location $\ell \in \overline{\mathcal{L}}_t \setminus \underline{\mathcal{L}}_t$ and computes $z_\ell^g\left(\mathcal{L}_t^{\text{in}}\right)$ and $z_\ell^g\left(\mathcal{L}_t^{\text{out}}\right)$. If $z_\ell^g\left(\mathcal{L}_t^{\text{in}}\right)$ is within the interval $\mathcal{Z}_t$, then the partition refines to include ℓ for all $z \in \mathcal{Z}_t$ above this cutoff; similarly, if $z_\ell^g\left(\mathcal{L}_t^{\text{out}}\right)$ is within the interval, then the partition refines to include ℓ for all $z \in \mathcal{Z}_t$ below this cutoff. The computational implementation refines each interval independently, and has converged when $A_t = \overline{\mathcal{L}}_t \setminus \underline{\mathcal{L}}_t$ for each interval.

Once generalized squeezing has converged, any interval for which $\underline{\mathcal{L}}_t \neq \overline{\mathcal{L}}_t$ is refined with iterative cutoff search, described in Section E.2. In particular, the computational implementation does not use generalized branching.

Finally, the computed policy function is returned as a series of cutoffs $\{z_t\}_{t=1}^{T+1}$ which define the intervals, together with the optimal decision sets for each interval $\{z_t\}_{t=1}^{T}$.

H.2. Computing and Calibrating the General Equilibrium Model

Aggregation The general equilibrium conditions require aggregating over the decisions of individual firms. Aggregation in practice is straightforward since the policy function $\mathcal{L}_i^\star(\cdot)$ for firms originating from i is simply a set of productivity intervals $\{[z_{i,t}, z_{i,t+1}]\}_t$ and their associated optimal decision sets $\left\{\mathcal{L}_{i,t}^\star\right\}_t$.

For example, consider the price index condition (5), which requires integrating the pricing decisions across all firms with positive production. Given the computed policy function, the condition can be rewritten as follows.

$$P_n^{1-\sigma} = \sum_i M_i \mu^{1-\sigma} \sum_{\mathcal{Z}_i^t \in \mathcal{T}_i} \left(\sum_{\ell \in \mathcal{L}_i^t} \xi_{i\ell n}^{-\theta} \right)^{\frac{\sigma-1}{\theta}} \int_{z_{i,t}}^{z_{i,t+1}} z^{\sigma-1} \mathrm{d}G_i\left(z\right).$$

In particular, the integral can be divided by the intervals. Since the optimal location set is constant within each interval, integration need only be performed over the firm types. Moreover, with firm productivity G_i following the Pareto distribution, this integral can be evaluated closed-form. Aggregation for the other equilibrium conditions follows a similar logic.

Solution loops The policy function computation is embedded in a larger computational routine to compute and estimate the general equilibrium of the model.

To compute the general equilibrium of the model, we start with an initial guess for the aggregates $\left\{P_\ell^{(0)}, w_\ell^{(0)}, M_\ell^{(0)}\right\}_\ell$, then use the following iterative routine.

1. Given aggregates $\left\{P_\ell^{(k)}, w_\ell^{(k)}, M_\ell^{(k)}\right\}_\ell$, directly set $X_\ell = w_\ell H_\ell$ for all ℓ to satisfy condition (7).
2. Solve the policy functions $\left\{\mathcal{L}_i^\star\left(\cdot\right)\right\}_i$ of the firm's CDCP from equation (2) using the policy function method. The policy function also determines $\tilde{z}_i$ as the lowest type to operate at least one location, satisfying condition (3).
3. Given the policy functions $\left\{\mathcal{L}_i^\star\left(\cdot\right)\right\}_i$, use deviations from aggregate conditions to update the aggregates. In particular,
   - update $\left\{P_\ell^{(k)}\right\}_\ell$ using (5);
   - update $\left\{w_\ell^{(k)}\right\}_\ell$ using (6); and
   - update $\left\{M_\ell^{(k)}\right\}_\ell$ using (4).

To estimate the general equilibrium model, we set the parameters $\{\sigma, \theta, \zeta\}$ as well as country-level variables $\{H_\ell, w_\ell\}_\ell$ as described in Table 1. We also set $X_\ell = w_\ell H_\ell$ following condition (7) as well as the number of entrants $\{M_i\}_i$ by dividing the empirical number of enterprises by the empirical survival rate.

We then iteratively calibrate, holding fixed these values. We start with an initial guess for the market size aggregate $\left\{P_n^{(0)}\right\}_n$, the fundamentals $\left\{T_\ell^{(0)}, \underline{z}_\ell^{(0)}, f_\ell^{(0)}\right\}_\ell$, and the bilateral cost parameterization $\left\{\overline{\tau}_\ell^{(0)}, \overline{\gamma}_\ell^{(0)}, \overline{\nu}_\ell^{(0)}\right\}_\ell$ with $\left\{\kappa_\tau^{v(0)}, \kappa_\gamma^{v(0)}, \kappa_\nu^{v(0)}\right\}_v$. Each iteration then proceeds as follows.

1. Solve the policy functions $\left\{\mathcal{L}_i^\star\left(\cdot\right)\right\}_i$ of the firm's CDCP from equation (2) using the policy

function method. The policy function also determines $\tilde{z}_i$ as the lowest type to operate at least one location, satisfying condition (3).

2. Given $\{\mathcal{L}_i^\star(\cdot)\}_i$, compute trade, MP, and affiliate flows in the model; estimate the PPML specification of Appendix C.2.
3. Use deviations from aggregate conditions and moments to update. In particular,
   - update $\left\{P_n^{(k)}\right\}_n$ using (5);
   - update $\left\{T_\ell^{(k)}, \underline{z}_\ell^{(k)}, f_\ell^{(k)}\right\}_\ell$ using respectively deviations from (6), the gap between model and data total foreign MP conducted by firms from each country, and the gap between model and data survival rate in each country;
   - update $\left\{\overline{\tau}_\ell^{(k)}, \overline{\gamma}_\ell^{(k)}, \overline{\nu}_\ell^{(k)}\right\}_\ell$ using the gap between model and data own shares of trade (absorption), MP (domestic production), and affiliates (domestic production locations); and
   - update $\left\{\kappa_\tau^{v(k)}, \kappa_\gamma^{v(k)}, \kappa_\nu^{v(k)}\right\}_v$ using the gap between the model and data coefficients of the PPML regression.

The iteration converges to (near) zero on all deviations. The routine thus computes the GE and calibrates the model at once. After convergence, invert the aggregate condition (4) to recover f_i^e.

I. Additional Calibration and Counterfactual Details

In this section, we present additional figures and tables that accompany Sections 4 and 5.

Estimated Fundamentals In the left panels of Figure 10, we plot our estimates of the headquarter productivity shifter $\underline{z}_i$ of the firm Pareto distribution against the location productivity shifter T_ℓ of the Fréchet distribution of location-input-specific productivity shocks. Recall that $\underline{z}_i$ is identified by the total amount of foreign affiliate sales of firms headquartered in country i, while T_ℓ is chosen to match countries' level of GDP per capita. Unsurprisingly, the US has the highest headquarter productivity in our dataset. Relatively developed economies with little MNE activity, such as Greece and Spain, lie below the US and to the right of the 45 degree line that defines the US comparative advantage. On the opposite side of the 45 degree line and close to the US lie developed countries with relatively high MNE activity such as Netherlands, Germany, and Finland.

In the right panel of Figure 10, we plot the entry cost and the base component of the fixed cost for each country. In our data, one-year survival rates range from 77 to 93 percent, so that the ratio between the base component of the fixed cost and the entry cost varies little across countries. More importantly, there is a strong correlation between the level the base component

	Negative Complementarities				Positive Complementarities			
	Log MP Costs		Log Fixed Costs		Log MP Costs		Log Fixed Costs	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Log Distance	-0.06^{**}	0.00	0.75^{***}	0.35	-0.15^{***}	0.02	0.88^{***}	0.43
	(0.02)	(.)	(0.04)	(.)	(0.03)	(.)	(0.04)	(.)
Colony		0.03		-0.19		0.04		-0.21
		(.)		(.)		(.)		(.)
Contiguity		-0.06		-0.05		-0.13		0.01
		(.)		(.)		(.)		(.)
Language	-0.53^{***}	-0.07	0.70^{***}	-0.08	-0.81^{***}	-0.11	0.67^{***}	-0.20
	(0.07)	(.)	(0.14)	(.)	(0.10)	(.)	(0.15)	(.)
Destination FE	No	Yes	No	Yes	No	Yes	No	Yes
Origin FE	Yes	No	Yes	No	Yes	No	Yes	No
N	707	707	707	707	707	707	707	707

TABLE 4: CALIBRATED COSTS AND GRAVITY VARIABLES

The table shows results from regressing the log of the calibrated MP costs and fixed costs on log distance. and other gravity variables. The even-numbered columns are the true specification of costs in the model, reflected in the zero standard errors. The odd-numbered columns reflect the specification of Alviarez, Cravino, and Ramondo (2023). The left panel shows results from the baseline calibration while the right panel replicates these regressions in the alternative calibration with no fixed costs. We drop country pairs with zero MP for which we infer infinite MP costs and fixed costs. We also drop own country pairs. Levels of significance are denoted as follows: *** Significant at 1 percent level, ** Significant at 5 percent level, and * Significant at 10 percent level.

of the fixed cost and GDP of each country. This relationship is the result of an important pattern in our data: the log number of active enterprises increases with the log country population but less than one for one (coefficient of 0.81). To generate this pattern, our model requires fixed costs and entry costs that are increasing with population.[21] This empirical regularity is in contrast with many tractable models of monopolistic competition with firm entry in which the number of entrants is a linear function of the population, such as Melitz (2003), Chaney (2008), or Arkolakis, Ramondo, et al. (2018). Because our model does not imply this relationship, our calibration requires a direct measure of entering enterprises M_i, in addition to a measure of local population H_i, in line with Adão, Arkolakis, and Ganapati (2020).

Table 4 shows the relationship between the calibrated MP costs and fixed costs by distance and other gravity variables. We drop pairs of countries for which we observe no MP activity in the data since we infer infinite MP costs for them. We drop diagonal entries of the cost matrices which are normalized to 1.

[21] In a dynamic setup as in Melitz (2003), our estimates of firm entry costs correspond to the yearly amortized entry cost.

The table compares the results from the calibration with negative complementarities on the left against the calibration with positive complementarities on the right. Even-numbered columns represent the true specification of costs in the model, reflected in the zero standard errors. On the other hand, odd-numbered columns follow the specification of Alviarez, Cravino, and Ramondo (2023), in particular including origin-specific fixed effects but omitting the colony and contiguity gravity variables.

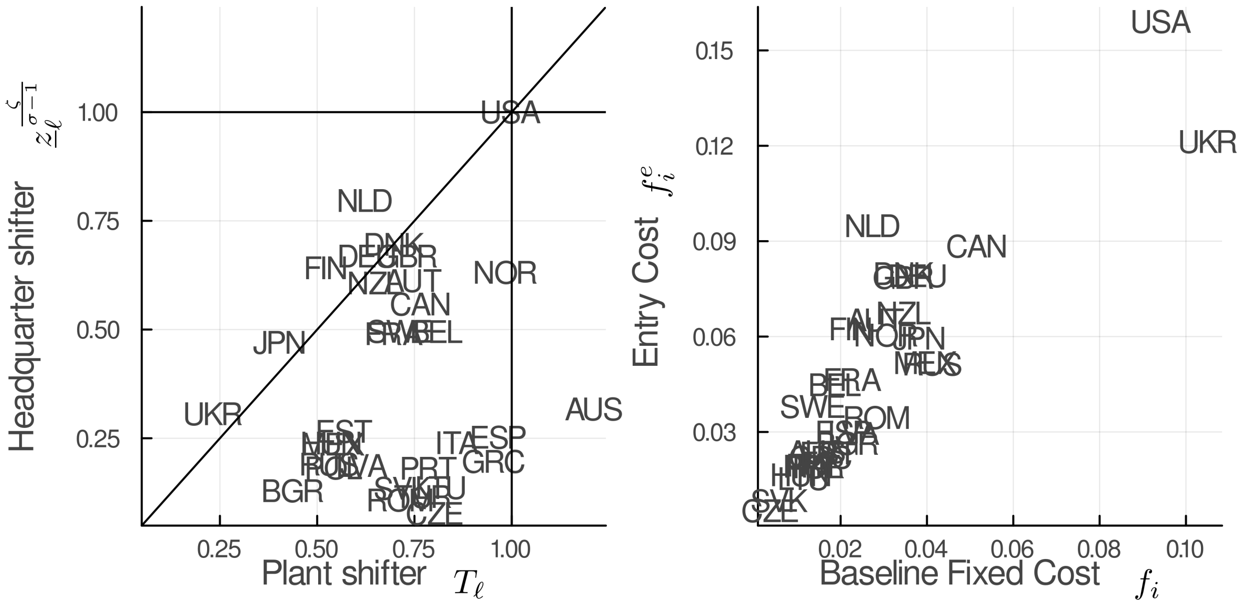
Figure 11 graphs the trade shares, inward MP sales shares, and inward foreign affiliate shares generated by the calibrated models against the same objects in the data. The model provides a good fit, especially for larger shares. The fit for larger shares is better since the targeted PPML specification in Table 3 is in levels, thereby putting relatively more weight on larger countries (see Sotelo 2019, for a discussion). The model-generated data produces exactly the same coefficient estimates as in Table 3.

Alternative Calibration without Fixed Costs Many papers in the multinational production literature abstract from modeling fixed costs, which simplifies computation by eliminating the CDCP (see, e.g., Ramondo and Rodríguez-Clare 2013; Ramondo 2014; Arkolakis, Ramondo, et al. 2018; Fajgelbaum et al. 2019). Without fixed costs, firms set up production locations in all countries and only face the intensive margin problem of choosing how much to produce in each. Therefore, to better understand the role of fixed costs in determining economic allocations, we also calibrate a version of the model where we set the fixed costs to zero, so that $f_i = 0$ for all origins i. As a result, the firm establishes production in all locations $\ell \in L$ so profit function collapses to

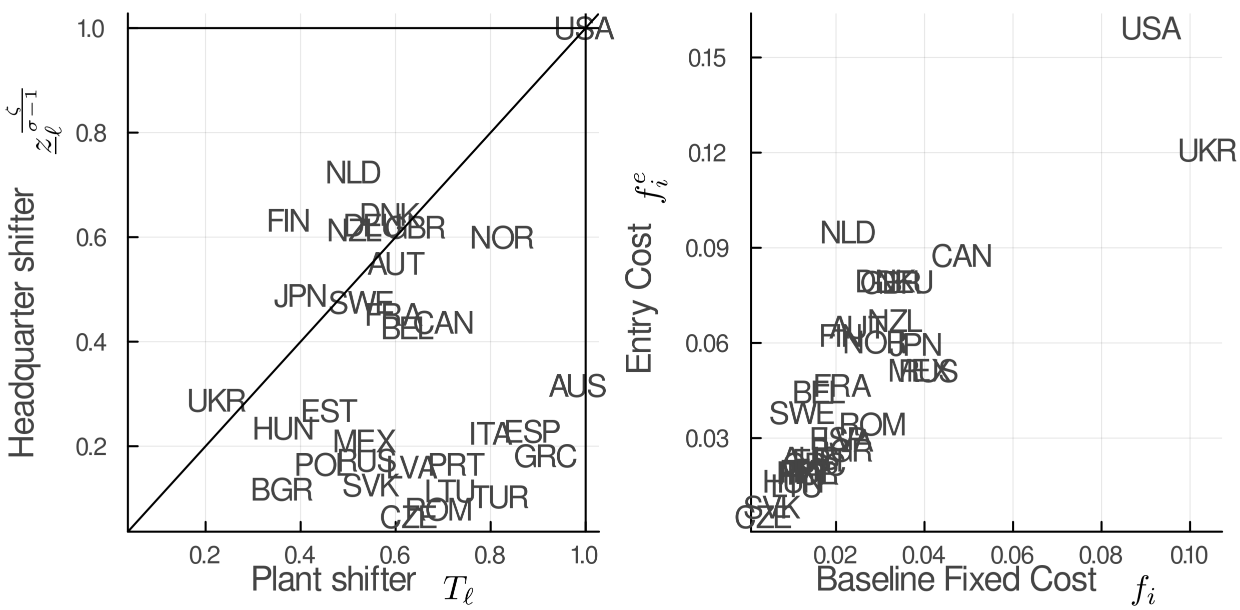
$$\pi_i(z) = \sum_n (\mu - 1)\, q_{in}(L, z)\, c_{in}(L, z)\ .$$

We then follow nearly the same procedure as with the full model, but drop as calibration targets the survival rate of firms and the coefficients in the third column of Table 3. When there are no fixed costs of operation or production, all entrants survive and establish production in all locations ℓ. We compare results from this calibration to our baseline calibration in Section 5.2, to understand the importance of fixed costs in shaping counterfactual responses of the economy to economic shocks.

Figure 12 reports the welfare impact of moving from the baseline calibrated economy to MP autarky, for the benchmark calibrations with fixed costs as well as the alternative calibrations without fixed costs. For most countries, the welfare consequence of MP autarky is more negative in the calibration without fixed costs compared to the benchmark calibration with fixed costs.



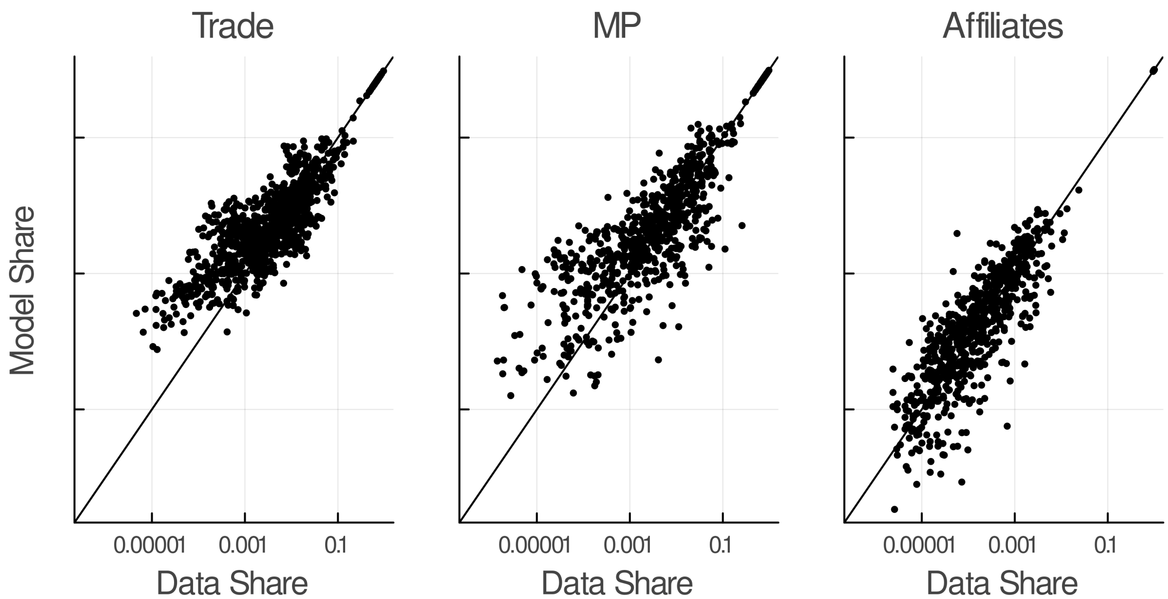
(A) Negative Complementarities



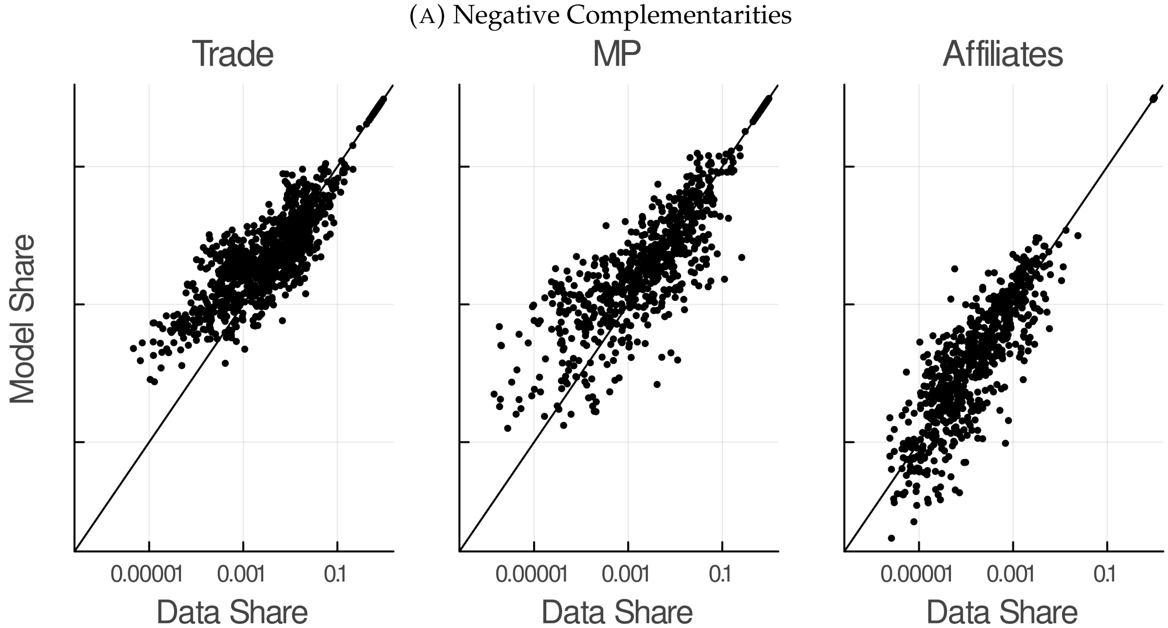
(B) Positive Complementarities

FIGURE 10: TECHNOLOGY, BASE COMPONENT OF FIXED COSTS, AND ENTRY COSTS IN THE BASELINE CALIBRATION

The figure shows a number of calibrated shifters in the model. The left panel graphs the Pareto minimum $\underline{z}_i^{\xi/(\sigma-1)}$ of the firm productivity distribution against the scale T_ℓ of Fréchet distribution of location-input-specific productivity shocks. The terms $\underline{z}_i^{\xi/(\sigma-1)}$ and T_ℓ appear multiplicatively in the expression for trilateral flows. The right panel plots the entry cost f_i^e against the base component of the fixed cost f_i.



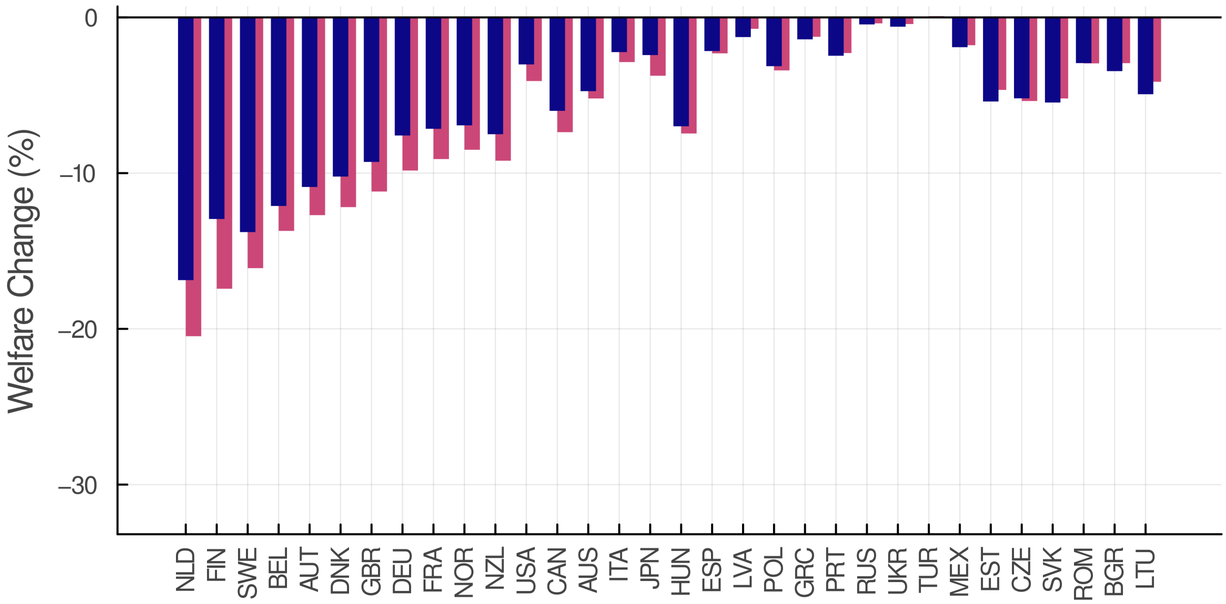
(A) Negative Complementarities



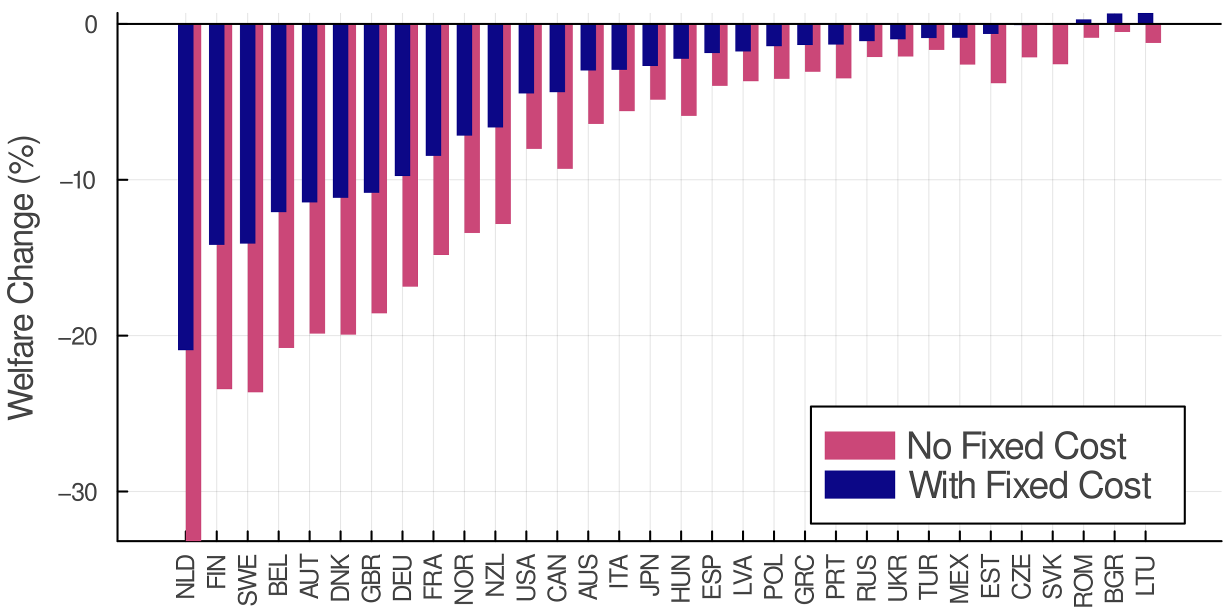
(B) Positive Complementarities

FIGURE 11: TRADE SHARES, INWARD MP SALES SHARES, AND INWARD AFFILIATE SHARES IN THE DATA AND THE BASELINE CALIBRATION

The figures shows graphs statistics from the data obtained from Alviarez (2019) against the same objects in the calibrated model. The left panel shows trade shares, the second panel shows inward MP sales shares, and the third panel shows inward MP affiliate shares. The correlations between the off-diagonal shares in the model and data are 0.757, 0.726, and 0.824 respectively in the calibration with negative complementarities; and 0.802, 0.763, 0.728 respectively in the calibration with positive complementarities.



(A) Negative Complementarities



(B) Positive Complementarities

FIGURE 12: WELFARE CONSEQUENCES OF SHUTTING DOWN MULTINATIONAL PRODUCTION WITH AND WITHOUT FIXED COSTS

The figure shows the percentage welfare change $100 \times \left(\hat{w}_i / \hat{P}_i - 1\right)$ from moving from the calibrated economy to MP autarky. The countries are ordered by the size of the welfare effect in the calibration with positive complementarities and fixed costs.