Combinatorial Discrete Choice

Teaching slides

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Motivation

- Discrete choice problems with complementarities among options
  - Tesla choosing in which countries to operate production plants
  - Starbucks choosing blocks in Manhattan to operate shops
  - A government choosing locations for critical infrastructure
- Without more structure: an intractable NP hard problem
- This paper. Solve such combinatorial discrete choice problems
- Key. Economic complementarities provide exploitable structure

Part I

Theory

Notation

- Set of discrete options L
  Index individual items in L by ℓ, so that $\ell \in L$
- Define collection of subsets (power set) of L as: $\mathscr{P}(L)$
  Denote individual element in $\mathscr{P}(L)$ by $\mathscr{L}$, so that $\mathscr{L} \in \mathscr{P}(L)$
- Define the space of objective functions $\mathscr{F} = \{f : \mathscr{P}(L) \to \mathbb{R}\}$
  Denote an individual objective by f, so that $f \in \mathscr{F}$

Outline

Characterization

- Maximization over subsets. Choose the subset of items $\mathscr{L} \subseteq L$

  leading example: multinational location problem

$$\mathscr{L}^\star = \arg\max_{\mathscr{L} \subseteq L} f(\mathscr{L})$$

- Marginal value operator. For an item ℓ, the value with it compared to without it, contingent on $\mathscr{L}$

  discrete analogue to derivative

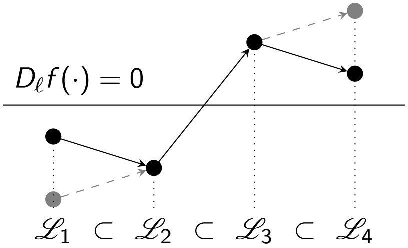
$$D_\ell f(\mathscr{L}) = f(\mathscr{L} \cup \{\ell\}) - f(\mathscr{L} \setminus \{\ell\})$$

- *Combinatorial* discrete choice. If the marginal value varies with $\mathscr{L}$

Single crossing differences in choices

From below. If ℓ is valuable given a small set, *remains* valuable given a large set:

$$D_\ell f\left(\mathscr{L}\right) \geq 0 \quad \Rightarrow \quad D_\ell f\left(\mathscr{L}'\right) \geq 0$$



Supermodularity. *More* valuable given large set compared to small set

$$D_\ell f\left(\mathscr{L}\right) \leq D_\ell f\left(\mathscr{L}'\right)$$

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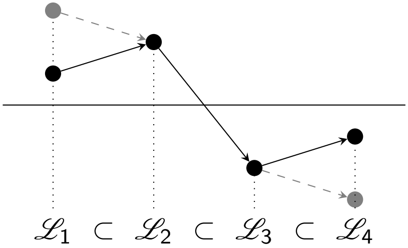


Submodularity. *More* valuable given small set compared to large set

$$D_\ell f\left(\mathscr{L}\right) \geq D_\ell f\left(\mathscr{L}'\right)$$

Single crossing differences in choices

Definition (Quasi-supermodularity and quasi-submodularity)

The function f is:

a) *quasi-supermodular if, for all $\mathscr{L}, \mathscr{L}' \in \mathscr{P}(L)$,*

$$f(\mathscr{L} \cup \mathscr{L}') \leq f(\mathscr{L}') \qquad \Rightarrow \qquad f(\mathscr{L}) \leq f(\mathscr{L} \cap \mathscr{L}')$$

b) *quasi-submodular if, for all $\mathscr{L}, \mathscr{L}' \in \mathscr{P}(L)$,*

$$f(\mathscr{L}) \geq f(\mathscr{L} \cap \mathscr{L}') \qquad \Rightarrow \qquad f(\mathscr{L} \cup \mathscr{L}') \geq f(\mathscr{L}')$$

Shannon and Milgrom 1994; Milgrom 2004

Corollary

Quasi-supermodularity is sufficient for SCD-C from below; quasi-submodularity is sufficient for SCD-C from above.

Our single-crossing in choices condition is closely related to the quasi-supermodularity and quasi-submodularity conditions from the monotone comparative statics literature, pioneered in Shannon and Milgrom 1994.

Proof.

We show the statements for quasi-submodularity and SCD-C from above. A similar argument follows for quasi-supermodularity and SCD-C from below.
Suppose f is quasi-submodular and let $\mathscr{L} \in \mathscr{P}(L)$, $\ell \in L$ with $D_\ell f(\mathscr{L}) \geq 0$. Select any $\mathscr{L}' \subseteq \mathscr{L}$. We show that $D_\ell(\mathscr{L}') \geq 0$. Let $\mathscr{J} \equiv \mathscr{L}' \cup \{\ell\}$ and $\mathscr{K} \equiv \mathscr{L} \setminus \{\ell\}$. Then,

$$\begin{aligned} D_\ell f(\mathscr{L}) &= f(\mathscr{L} \cup \{\ell\}) - f(\mathscr{L} \setminus \{\ell\}) \\ &= f(\mathscr{J} \cup \mathscr{K}) - f(\mathscr{K}) \geq 0 \\ \Rightarrow f(\mathscr{J}) &\geq f(\mathscr{J} \cap \mathscr{K}) \end{aligned}$$

where the last line follows from quasi-submodularity. Then, it follows that $D_\ell(\mathscr{L}') \geq 0$. □

"Local optimality"

- Jia 2008. Central mapping:

$$\Phi(\mathscr{L}) = \{\ell \in L \mid D_\ell f(\mathscr{L}) \geq 0\}$$

"All items with non-negative marginal value to $\mathscr{L}$"

- No deviation by one element. Necessary, not sufficient!

similar to a first order condition

$$\mathscr{L}^\star = \Phi(\mathscr{L}^\star)$$

  - if ℓ is chosen ($\ell \in \mathscr{L}^\star$), then it must contribute positive marginal value ($\ell \in \Phi(\mathscr{L}^\star)$)
  - if ℓ is *not* chosen ($\ell \notin \mathscr{L}^\star$), then it cannot add value when included ($\ell \notin \Phi(\mathscr{L}^\star)$)

Order-preserving (reversing)

Lemma

If f satisfies SCD-C from below (above), Φ is order-preserving (reversing).

Proof.

We show for SCD-C from above and order-reversing. Let $\mathcal{L} \subset \mathcal{L}'$ be two arbitrary nested decision sets.

Start with the converse. Suppose f obeys SCD-C from above. If $\Phi(\mathcal{L}')$ is empty, then it is contained in $\Phi(\mathcal{L})$ trivially; so let $\ell \in \Phi(\mathcal{L}')$ be an arbitrary element. Then, by definition of ,$\mathrm{D}_\ell f(\mathcal{L}) \geq 0$. With SCD-C from above, it must be that $D_\ell f(\mathcal{L}') \geq 0$; hence, $\ell \in \Phi(\mathcal{L})$. Then, $\Phi(\mathcal{L}') \subseteq \Phi(\mathcal{L})$ and Φ is order-reversing.

Now consider the forward direction. Let ℓ be an arbitrary element so that $D_\ell f(\mathcal{L}') \geq 0$. If no such element exists, then SCD-C from above holds vacuously, so suppose at least one such ℓ exists. Then, by definition, $\ell \in \Phi(\mathcal{L}') \subseteq \Phi(\mathcal{L})$ since Φ is order-reversing. Then, by definition of Φ, it must be that $D_\ell f(\mathcal{L}) \geq 0$.

A reverse argument holds for SCD-C from below. □

Squeezing mapping

- Bounding pair $\left[\underline{\mathscr{L}}, \overline{\mathscr{L}}\right]$. Defines a restricted domain

$$\left\{\mathscr{L} \middle| \underline{\mathscr{L}} \subseteq \mathscr{L} \subseteq \overline{\mathscr{L}}\right\} \subseteq \mathscr{P}(L)$$

  - the full domain is represented $[\emptyset, L] = \mathscr{P}(L)$
  - $\left[\underline{\mathscr{K}}, \overline{\mathscr{K}}\right]$ is "tighter" than $\left[\underline{\mathscr{L}}, \overline{\mathscr{L}}\right]$ if $\left[\underline{\mathscr{K}}, \overline{\mathscr{K}}\right] \subseteq \left[\underline{\mathscr{L}}, \overline{\mathscr{L}}\right]$, i.e. it defines a subdomain
- Squeezing mapping. Acts on bounding pairs

$$S\left(\left[\underline{\mathscr{L}}, \overline{\mathscr{L}}\right]\right) = \left[\inf\left\{\Phi\left(\underline{\mathscr{L}}\right), \Phi\left(\overline{\mathscr{L}}\right)\right\}, \sup\left\{\Phi\left(\underline{\mathscr{L}}\right), \Phi\left(\overline{\mathscr{L}}\right)\right\}\right]$$

- Iterative application. Let $S^k\left(\left[\underline{\mathscr{L}}, \overline{\mathscr{L}}\right]\right)$ denote applying S iteratively k times

Main theorem: Single agent problem

Theorem 1 (Squeezing procedure)

If f satisfies SCD-C, then:

a. *let* $\left[\underline{\mathscr{L}}^{(k)}, \overline{\mathscr{L}}^{(k)}\right] \equiv S^k\left([\emptyset, L]\right)$; *then,*

$$\emptyset \subseteq \ldots \subseteq \underline{\mathscr{L}}^{(k)} \subseteq \underline{\mathscr{L}}^{(k+1)} \subseteq \overline{\mathscr{L}}^{(k+1)} \subseteq \overline{\mathscr{L}}^{(k)} \subseteq \ldots \subseteq L$$

"iterative application weakly tightens the problem's domain"

b. *if* $\mathscr{L}^\star \in [\mathscr{L}, \mathscr{L}']$, *then* $\mathscr{L}^\star \in S\left([\mathscr{L}, \mathscr{L}']\right)$

"if the optimum set is in the restricted domain, S will not discard it"

c. $S^{|L|}\left([\emptyset, L]\right) = S^{|L|+1}\left([\emptyset, L]\right)$

"iterating the squeezing step S converges to a fixed point in $|L|$ steps or fewer"

Proof.

a. The proof follows by induction on k:

- When $k = 1$, the statement $\emptyset = \underline{\mathscr{L}}^{(0)} \subseteq \underline{\mathscr{L}}^{(1)} \subseteq \overline{\mathscr{L}}^{(1)} \subseteq \overline{\mathscr{L}}^{(0)} = L$ holds vacuously.
- Inductive step: suppose, for k, the statement holds.
- Consider iteration $k+1$. From the inductive step, we have $\underline{\mathscr{L}}^{(k-1)} \subseteq \underline{\mathscr{L}}^{(k)} \subseteq \overline{\mathscr{L}}^{(k)} \subseteq \overline{\mathscr{L}}^{(k-1)}$. Applying Φ and using the lemma,

$$\begin{cases} \Phi\left(\overline{\mathcal{L}}^{(k-1)}\right) \subseteq \Phi\left(\overline{\mathcal{L}}^{(k)}\right) \subseteq \Phi\left(\underline{\mathcal{L}}^{(k)}\right) \subseteq \Phi\left(\underline{\mathcal{L}}^{(k-1)}\right) & \text{if SCD-C from above} \\ \Phi\left(\underline{\mathcal{L}}^{(k-1)}\right) \subseteq \Phi\left(\underline{\mathcal{L}}^{(k)}\right) \subseteq \Phi\left(\overline{\mathcal{L}}^{(k)}\right) \subseteq \Phi\left(\overline{\mathcal{L}}^{(k-1)}\right) & \text{if SCD-C from below} \end{cases}$$

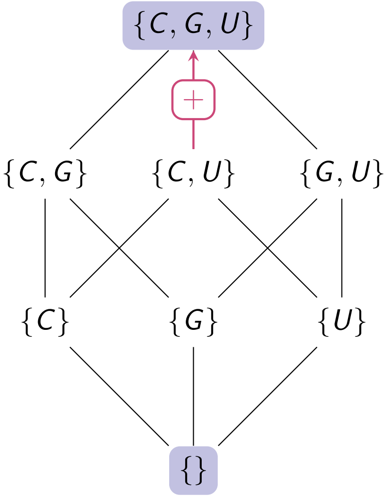
$$\Rightarrow \underline{\mathcal{L}}^{(k)} \subseteq \underline{\mathcal{L}}^{(k+1)} \subseteq \overline{\mathcal{L}}^{(k+1)} \subseteq \overline{\mathcal{L}}^{(k)}$$

b. Follows from Φ order-preserving (reversing) and $\Phi(\mathscr{L}^\star) = \mathscr{L}^\star$.

c. Each (non-trivial) application, S must add one element to $\underline{\mathscr{L}}$ or remove one element from $\overline{\mathscr{L}}$; there are $|L|$ total elements.

□

Proof intuition: SCD-C from above



- Bounding pair. $\underline{\mathscr{L}} \subseteq \mathscr{L}^\star \subseteq \overline{\mathscr{L}}$
  - $\underline{\mathscr{L}}$ tracks elements in $\mathscr{L}^\star$
  - $\overline{\mathscr{L}}$ discards elements not in $\mathscr{L}^\star$
  - $\overline{\mathscr{L}} \setminus \underline{\mathscr{L}}$ tracks elements maybe in $\mathscr{L}^\star$
- Rule out suboptimal strategies.
  - check marginal value at points of extreme complementarity
  - iteratively squeeze: update the subset and superset

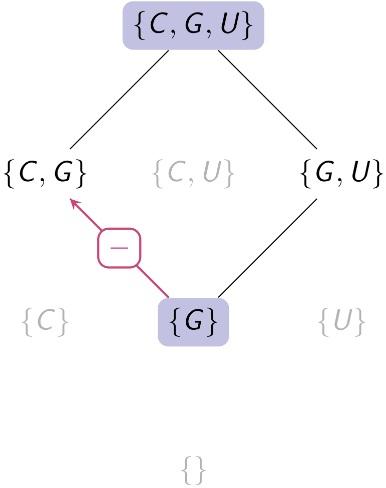
    halve decision space each time

Each step in this example pins down an additional country:

1. When all countries are active, G has positive marginal value $\rightarrow$ It must also have positive marginal value for any smaller set $\mathscr{L} \subseteq \{C, G, U\}$.
   - G must be in the optimal set
   - update $\underline{\mathcal{L}}$ to include G
     discards any decision set that doesn't include G (half of the remaining decision sets)
2. Given the smallest possible decision set $\{G\}$, the C country still has negative value.
   - it cannot be in the optimal set
   - update $\overline{\mathcal{L}}$ to discard C
     discards any remaining decision set that includes C
3. Given the largest possible decision set $\{G, U\}$, the U country has positive marginal value — it must be included in $\mathscr{L}^\star$

Through squeezing, we identify the optimal decision set $\mathscr{L}^\star = \{G, U\}$.

Proof intuition: SCD-C from above



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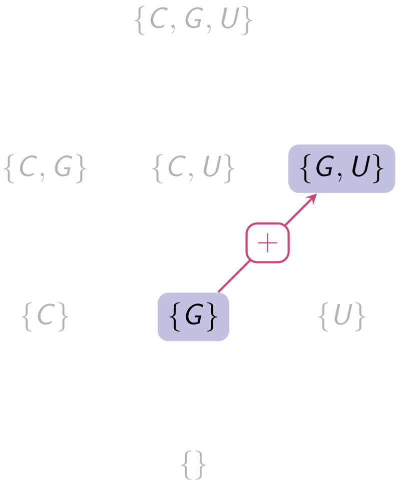

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Proof intuition: SCD-C from above

$\{C, G, U\}$

$\{C, G\}$ $\{C, U\}$ $\{G, U\}$

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$\{\}$

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SCD-C from below

Lattice foundation

- Jia 2008. Solution method for supermodular f:
  1. Central mapping. By construction, $\mathscr{L}^\star$ is a fixed point of:
  $$\Phi\left(\mathscr{L}\right) \equiv \left\{\ell \in L \mid D_\ell f\left(\mathscr{L}\right) \geq 0\right\}$$
  2. Order-preserving Φ. With supermodular f
  3. Tarski 1955. Order-preserving Φ has a smallest and largest fixed point ...
  4. Kleene 1936. ... identified by iterating $\Phi^\infty\left(\emptyset\right)$ and $\Phi^\infty\left(L\right)$
- SCD-C (from below). Necessary and sufficient condition for Φ to be order-preserving

SCD-C from above

Lattice foundation

- Order-reversing Φ. Tarski 1955; Kleene 1936 no longer apply
- Perfect substitutes intuition. Consider two elements, $\{a, b\}$
  - both items have positive marginal value in isolation, but neither have positive marginal value if the other is included

$$\Phi(\emptyset) = \{a, b\} \qquad \Phi(\{a, b\}) = \emptyset$$

  - the fixed points are uncomparable, i.e. there is neither a smallest nor largest fixed point — Tarski 1955 breaks down ...

$$\Phi(\{a\}) = \{a\} \qquad \Phi(\{b\}) = \{b\}$$

  - ...without the existence of smallest and largest fixed points, does iteration converge? To what?

SCD-C from above

Lattice foundation

A generalization of the notion of a fixed point:

Definition (Fixed edge)

Two sets, $\mathscr{L}$ and $\mathscr{L}'$ with

$$\Phi\left(\mathscr{L}\right) = \mathscr{L}' \qquad , \qquad \Phi\left(\mathscr{L}'\right) = \mathscr{L}$$

- **Klimeš 1981.** Order-reversing Φ has an "extreme" fixed edge $\mathscr{L}^{\text{inf}}, \mathscr{L}^{\text{sup}}$!

$$\mathscr{L}^{\text{inf}} \subseteq \mathscr{L} \subseteq \mathscr{L}' \subseteq \mathscr{L}^{\text{sup}}$$

- **Iteration.** $\lim_{n\to\inf} \Phi^{2n}\left(\emptyset\right) = \mathscr{L}^{\text{inf}}$ and $\lim_{n\to\inf} \Phi^{2n+1}\left(\emptyset\right) = \mathscr{L}^{\text{sup}}$
  vice versa from L

SCD-C from above

Lattice foundation

- Φ's "Fixed edge convergence". After enough applications, the mapping Φ alternates back and forth between the two points in the fixed edge
- Squeezing step. Converges to fixed point by construction:

$$S\left(\left[\mathscr{L}^{\mathsf{inf}}, \mathscr{L}^{\mathsf{sup}}\right]\right) = \left[\Phi\left(\mathscr{L}^{\mathsf{sup}}\right), \Phi\left(\mathscr{L}^{\mathsf{inf}}\right)\right] = \left[\mathscr{L}^{\mathsf{inf}}, \mathscr{L}^{\mathsf{sup}}\right]$$

by "flipping" the order of the two sets

Refinement: branching

- If $\mathscr{L}^{\text{inf}} = \mathscr{L}^{\text{sup}}$, then $\mathscr{L}^{\text{inf}} = \mathscr{L}^{\star}$
- Sometimes: converge, but $\mathscr{L}^{\text{inf}} \subset \mathscr{L}^{\star}$
  e.g. when complementarities very strong

$\{C, G, U\}$

$\{C, G\}$ $\{C, U\}$ $\{G, U\}$

$\{C\}$ $\{G\}$ $\{U\}$

$\{\}$

Details

To handle cases where the converged bounding pair doesn't identify the optimum, we develop and characterize the branching procedure.

Refinement: branching

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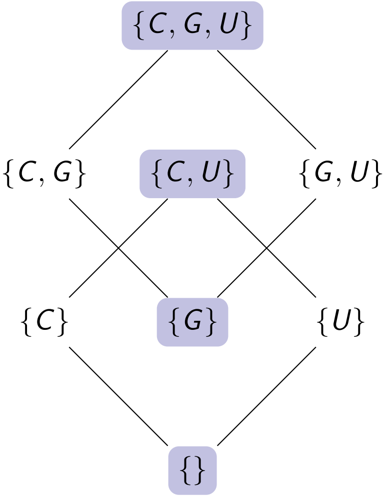
- Choose an item $\ell \in \overline{\mathscr{L}} \setminus \underline{\mathscr{L}}$, then
  - divide into two subproblems: with and without ℓ
  - squeeze on each problem, branching as needed tree



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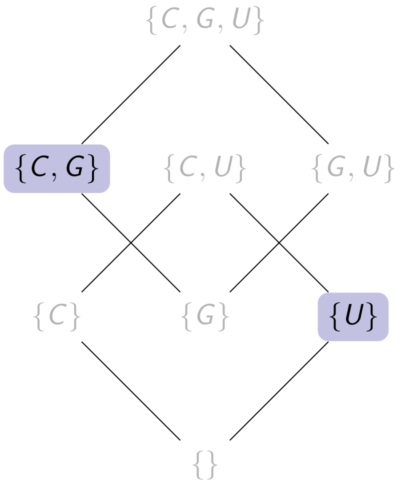
- Choose an item $\ell \in \overline{\mathscr{L}} \setminus \underline{\mathscr{L}}$, then
  - divide into two subproblems: with and without ℓ
  - squeeze on each problem, branching as needed tree
- End: "conditionally optimal" decision sets
  - among them, the global optimum
  - intuition: "brute force" one decision at a time, squeeze as much as possible



Outline

Heterogeneous agent problem

- Augmented objective function. $f : \mathscr{P}(L) \times \mathbb{R} \to \mathbb{R}$ maps the set $\mathscr{L}$ and the agent type $z \in \mathbb{R}$ to a scalar payoff $f(\mathscr{L}, z)$
  leading example: multinational location problem with heterogeneous productivity
- Policy function. Function $\mathscr{L}^\star : \mathbb{R} \to \mathscr{P}(L)$ specifies the optimal decision set for each type z:

$$\mathscr{L}^\star(z) = \arg \max_{\mathscr{L} \in \mathscr{P}(L)} f(\mathscr{L}, z)$$

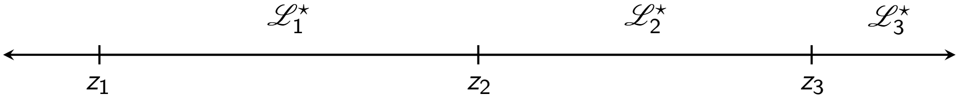
Single crossing differences in types

- **SCD in types (SCD-T).** For all elements $\ell \in L$, decision sets $\mathscr{L} \in \mathscr{P}(L)$, and types $z, z' \in \mathbb{R}$ such that $z < z'$,

$$D_\ell f(\mathscr{L}, z) \geq 0 \qquad \Rightarrow \qquad D_\ell f(\mathscr{L}, z') \geq 0$$

  SCD-T is equivalent to the single-crossing differences condition of Milgrom 2004 (originally "single crossing" in Shannon and Milgrom 1994).

- **With SCD-C and SCD-T.** The policy function changes its value only at a finite number of cutoff productivities:



- **Approach.** Partition type space into intervals that share the same policy; and find policy associated with each interval

If f satisfies *supermodularity* and SCD-T, Shannon and Milgrom 1994 show the policy function has a nesting structure:

$$z < z' \qquad \Rightarrow \qquad \mathscr{L}^{\star}(z) \subseteq \mathscr{L}^{\star}(z')$$

i.e. "higher productivity firms choose all locations selected by lower productivity firms and possibly more"

Type space partition

- Bounding set functions. Extend bounding pair to set-valued functions $\underline{\mathscr{L}}(\cdot)$ and $\overline{\mathscr{L}}(\cdot)$ with

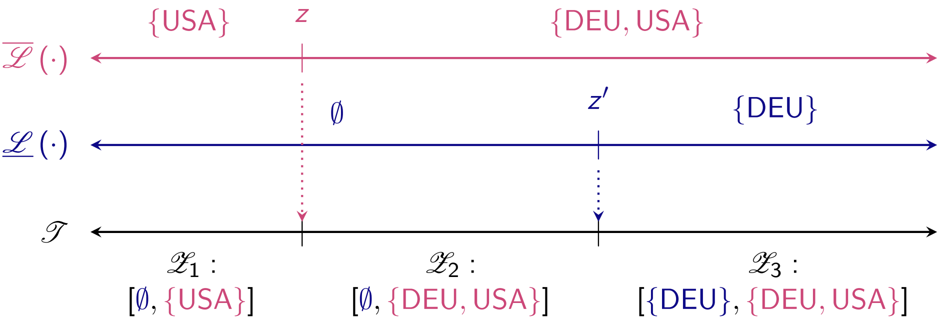
$$\underline{\mathscr{L}}(z) \subseteq \mathscr{L}^{\star}(z) \subseteq \overline{\mathscr{L}}(z)$$

for any productivity $z \in \mathbb{R}$
trivial bounding set functions: constant functions $[\emptyset, L]$

- Induced partition. From bounding set functions:

$$\mathscr{T}\left(\left[\underline{\mathscr{L}}(\cdot), \overline{\mathscr{L}}(\cdot)\right]\right) = \{\mathscr{Z}_1, \ldots \mathscr{Z}_t, \ldots \mathscr{Z}_T\}$$
$$\text{such that } \mathscr{Z}_t = \left\{z \in \mathbb{R} \mid \underline{\mathscr{L}}(z) = \underline{\mathscr{L}}_t, \overline{\mathscr{L}}(z) = \overline{\mathscr{L}}_t\right\},$$

Type space partition



Together, the two set-valued functions imply the partitioning $\mathscr{T}$, which creates intervals of productivity.

Details

- The top line illustrates an example upper bounding set function, while the middle illustrates an example lower bounding set function
- In this figure, there are three intervals, so $\mathscr{T} = \{\mathscr{Z}_1, \mathscr{Z}_2, \mathscr{Z}_3\}$.
- All productivities within a interval $\mathscr{Z}_t \in \mathscr{T}$ share the listed bounding pair.

Identifying cutoffs: intuition

- SCD-C. "Choice monotonicity" rules out decision sets without explicitly evaluating their payoff; together with . . .
- SCD-T. "Type monotonicity" means choice monotonicity can discard decision sets for productivity ranges without evaluating the objective at any of the productivities

Generalized squeezing

- With SCD-C + SCD-T. For each ℓ and $\mathscr{L}$, there is s unique type indifferent between including ℓ in $\mathscr{L}$

$$0 = D_{\ell}\left(\mathscr{L}, z_{\ell}^{g}\left(\mathscr{L}\right)\right)$$

- Indifferent type. Use to avoid evaluating $\Phi(\mathscr{L}, z)$ at each z for a given $\mathscr{L}$:

$$\Phi^{g}\left(\mathscr{L}, z\right) = \left\{\ell \mid z \geq z_{\ell}^{g}\left(\mathscr{L}\right)\right\}$$

- Generalized squeezing mapping.

$$S^{g}\left(\left[\mathscr{L}\left(\cdot\right), \mathscr{L}'\left(\cdot\right)\right]\right) = \Big[\inf\left\{\Phi^{g}\left(\mathscr{L}\left(\cdot\right), \cdot\right), \Phi^{g}\left(\mathscr{L}'\left(\cdot\right), \cdot\right)\right\}, \\ \sup\left\{\Phi^{g}\left(\mathscr{L}\left(\cdot\right), \cdot\right), \Phi^{g}\left(\mathscr{L}'\left(\cdot\right), \cdot\right)\right\}\Big]$$

Main theorem: Policy function

Theorem 2 (Generalized squeezing procedure)

If f satisfies SCD-C and SCD-T,

a. *Theorem 1a. and 1b. hold at each z*
b. $(S^g)^{|L|}([\emptyset, L]) = (S^g)^{|L|+1}([\emptyset, L])$

Proof.

Let $\Phi(\mathscr{L}, z) \equiv \{\ell \mid D_\ell f(\mathscr{L}, z) \geq 0\}$ be the mapping Φ evaluated at the type z. Applying Theorem 1 element-wise, we have for all z that

$$\underline{\mathscr{L}}_t \subseteq \Phi\left(\overline{\mathscr{L}}_t, z\right) \subseteq \mathscr{L}^\star(z) \subseteq \Phi\left(\underline{\mathscr{L}}_t, z\right) \subseteq \overline{\mathscr{L}}_t .$$

Then, it suffices to show that, for all z, $\Phi^g(\mathscr{L}, z)$ coincides with $\Phi(\mathscr{L}, z)$. The proof uses SCD-C and SCD-T to establish this equivalence. □

We prove the case of SCD-C from above. Select an arbitrary interval from the partition $\mathscr{Z}_t$; let its bounding pair be $\left[\underline{\mathscr{L}}_t, \overline{\mathscr{L}}_t\right]$. Choose an arbitrary type $z \in \mathscr{Z}_t$ and let $\ell \in \Phi^g\left(\overline{\mathscr{L}}_t, z\right)$ be an arbitrary element.

- Then, $z_\ell^g\left(\overline{\mathscr{L}}_t\right) < z$ and $0 = D_\ell f\left(\overline{\mathscr{L}}_t, z_\ell^g\left(\overline{\mathscr{L}}_t\right)\right)$ together imply $0 \leq D_\ell\left(\overline{\mathscr{L}}_t, z\right)$ by SCD-T. Thus, $\ell \in \Phi\left(\overline{\mathscr{L}}_t, z\right)$. Since ℓ was an arbitrarily chosen element of $\Phi^g\left(\underline{\mathscr{L}}_t, z\right)$, it follows that $\Phi^g\left(\underline{\mathscr{L}}_t, z\right) \subseteq \Phi\left(\underline{\mathscr{L}}_t, z\right)$ for all $z \in \mathscr{Z}_t$.
- Similarly, if $\ell \in \Phi\left(\underline{\mathscr{L}}_t, z\right)$ is an arbitrary element, then $0 \leq D_\ell f\left(\underline{\mathscr{L}}_t, z\right)$ implies $z \geq z_\ell^g\left(\underline{\mathscr{L}}_t\right)$ by SCD-T. Thus, $\ell \in \Phi^g\left(\underline{\mathscr{L}}_t, z\right)$ so $\Phi\left(\underline{\mathscr{L}}_t, z\right) \subseteq \Phi^g\left(\underline{\mathscr{L}}_t, z\right)$. This argument establishes that $\Phi^g\left(\overline{\mathscr{L}}_t, z\right) = \Phi\left(\overline{\mathscr{L}}_t, z\right)$ while a similar argument establishes $\Phi^g\left(\underline{\mathscr{L}}_t, z\right) = \Phi\left(\underline{\mathscr{L}}_t, z\right)$.

Corollary

Suppose the function f satisfies SCD-T and let $\ell \in L$ and $\mathscr{L}, \mathscr{L}' \in \mathscr{P}(L)$ where $\mathscr{L} \subset \mathscr{L}'$.

1. *If f also satisfies SCD-C from above, then $z_\ell^g(\mathscr{L}) \geq z_\ell^g(\mathscr{L}')$.*
2. *If f also satisfies SCD-C from below, then $z_\ell^g(\mathscr{L}) \geq z_\ell^g(\mathscr{L}')$.*

Proof intuition: SCD-C from above

For a given interval $\mathcal{Z}_t \in \mathcal{T}$:

1. select $\ell \in \overline{\mathscr{L}}_t \setminus \overline{\mathscr{L}}_t$, compute the two cutoffs

$$z_\ell^g\left(\underline{\mathscr{L}}_t\right) \qquad \leq \qquad z_\ell^g\left(\overline{\mathscr{L}}_t\right)$$

2. update bounding set functions:
   - for all firms with productivity $z < z_\ell^g\left(\underline{\mathscr{L}}_t\right)$ in $\mathscr{Z}_t$, ℓ is not part of the optimal decision set: update upper bounding set function to $\overline{\mathscr{L}}_t \setminus \{\ell\}$ for these productivities
   - for all firms with productivity $z > z_\ell^g\left(\overline{\mathscr{L}}_t\right)$ in $\mathscr{Z}_t$, ℓ is in the optimal decision set: update lower bounding set function to $\underline{\mathscr{L}}_t \cup \{\ell\}$ for these productivities
3. repeat for all $\ell \in \overline{\mathscr{L}}_t \setminus \overline{\mathscr{L}}_t$
4. use new bounding set functions to update partition

Part II

Application: MNEs

Outline

Quantitative framework

- Multinational firm CDCP
- SCD-C and SCD-T in firm problem
- Policy function and aggregation

Solution at work

- Solution method's performance
- Quantitative counterfactual

A model of multinational activity

- Setup.
  - Firms are born in origin country with productivity $z \sim g(z)$
  - Each firm produces a differentiated variety, compete monopolistically
  - There are L potential production locations
- Firm problem overview.
  - CDCP. Firms choose production locations subject to complementarities among locations and fixed costs
  - Heterogeneity. Productivity differences $\rightarrow$ Firms choose different production location sets $\rightarrow$ MNEs arise endogenously
- Full GE. Endogenous wages, firm entry, ...

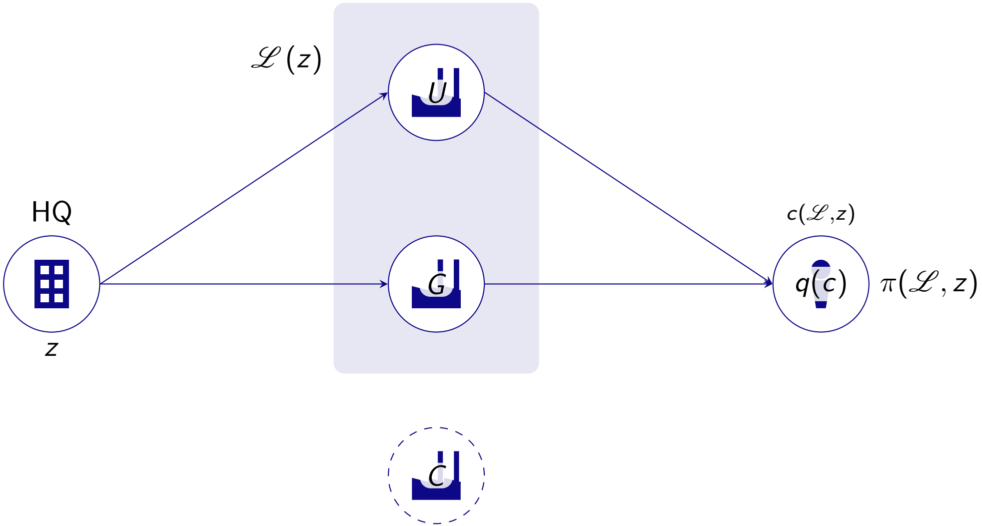
The firm problem

1. Location choice (extensive margin). Choose a set of production locations $\mathscr{L}$
   index origin country with i, production location with ℓ, destination market with n
2. Price and quantity (intensive margin). Choose price (quantity), contingent on CES marginal cost
   a possible microfoundation: input sourcing (Tintelnot 2017; Antràs, Fort, and Tintelnot 2017; Arkolakis, Ramondo, et al. 2018) details

$$c_{in}\left(\mathscr{L}, z\right) = \frac{1}{z}\left[\sum_{\ell \in \mathscr{L}} \left(\frac{w_\ell \gamma_{i\ell} \tau_{\ell n}}{T_\ell}\right)^{-\theta}\right]^{-\frac{1}{\theta}} = \frac{1}{z}\left[\sum_{\ell \in \mathscr{L}} \xi_{i\ell n}^{-\theta}\right]^{-\frac{1}{\theta}}$$

- marginal cost declines in $\mathscr{L}$
- θ: substitutability (complementarity) among locations in cost

The firm problem



CES marginal cost micro-foundation intution: details

- For each destination market n, the firm selects the production location $\ell \in \mathscr{L}$ that best serves the market
- The trilateral term $\xi_{i\ell n}$ thus bundles
  - bilateral friction between headquarter location i and production location ℓ
  - trade cost between production location ℓ and destination market n
  - production location ℓ cost of production
- locations are imperfectly substitutable (complementary) with elasticity θ

  at the extreme as $\theta \to \infty$, each market served by only one production location

Marginal cost, and thus profits, in each market depend on $\mathscr{L}$. In this example, the firm has production locations $\{U, G\}$ but not C.

The firm problem

- CES demand. In each market n,
  - the firm sets constant markup $\mu = \frac{\sigma}{\sigma-1}$ over marginal cost
  - let X_n be total expenditure and P_n be CES price index
- Total profits. Adding up over destination markets:

$$\pi_i(\mathscr{L}, z) \equiv \left[1 - \frac{1}{\mu}\right] \sum_n X_n \left(\frac{zP_n}{\mu}\right)^{\sigma-1} \left[\sum_{\ell \in \mathscr{L}} \xi_{i\ell n}^{-\theta}\right]^{\frac{\sigma-1}{\theta}} - \sum_{\ell \in \mathscr{L}} w_\ell f_{i\ell}$$

  where $f_{i\ell}$ is the fixed labor cost of establishing production in location ℓ
- Location choice policy function. The firm chooses production locations to maximize total profits:

$$\mathscr{L}_i^\star(z) = \arg\max_{\mathscr{L} \subseteq L} \pi_i(\mathscr{L}, z)$$

Firm location choice is a CDCP

- Marginal value of location k. Trades off the marginal cost savings with fixed cost:

$$\frac{1}{\sigma}\sum_n X_n \left(\frac{zP_n}{\mu}\right)^{\sigma-1} \left\{ \left[\xi_{ikn}^{-\theta} + \sum_{\mathscr{L}} \xi_{i\ell n}^{-\theta} \right]^{\frac{\sigma-1}{\theta}} - \left[\sum_{\mathscr{L}} \xi_{i\ell n}^{-\theta} \right]^{\frac{\sigma-1}{\theta}} \right\} - w_k f_{ik}$$

  complementarities preclude deciding on each location independently from the other locations

- Applying our solution. Establish SCD first:
  - SCD-C. Sufficient condition: $\sigma - 1 \lesseqgtr \theta$

    θ cost-side cannibalization (or complementarity)

    σ demand-side market-level scale effect

  - SCD-T. Sufficient condition: $\sigma > 1$

general demand

- Double CES setup. SCD-C and SCD-T map to simple conditions on parameters, which clearly demonstrate the forces that generate complementarities:
  - Demand side. Positive complementarity governed by demand elasticity σ:
    - indexes the firm's ability to scale
    - larger $\sigma \rightarrow$ demand is more sensitive to prices $\rightarrow$ marginal cost savings of large multinationals translates into even larger sales

    these returns to scale make each additional location more valuable
  - Cost side. Governed by elasticity θ:
    - $\theta > 1$. production locations are substitutes as they compete with one another to supply any destination ("cannibalization")
    - $0 < \theta < 1$. locations act as complements in production

  The firm CDCP's overall pattern of complementarity is the net effect.
- Generality. This overall intuition holds with more general demand and cost structures

  general demand

Policy function in practice

- Policy function $\mathscr{L}_i(z)$. Maps firm productivity z to production location set example
- Aggregation. Production in location ℓ of the average active firm from origin i requires integrating over optimal decision set for each active type z

$$\sum_n X_n \left(\frac{zP_n}{\mu}\right)^{\sigma-1} \int \frac{\mathbf{1}_\ell^\star(z)\xi_{i\ell n}^{-\theta}}{\sum_{k\in\mathscr{L}^\star(z)} \xi_{ikn}^{-\theta}} \left[\sum_{\mathscr{L}_i^\star(z)} \xi_{i\ell n}^{-\theta}\right]^{\frac{\sigma-1}{\theta}} \mathrm{d}G_i\left(z \mid \text{active}\right)$$

- Gravity at the firm level (across locations), but not in the aggregate

Closing and quantifying the model

- Aggregate conditions. details
  - Free entry with entry labor cost f_i^e
  - Labor market clearing with H_ℓ units, inelastically supplied
  - Balance of payments
- Quantification. Calibrate with:
  - 32 countries using aggregate data details
    Alviarez 2019; Feenstra, Inklaar, and Timmer 2015
  - two levels of complementarities to highlight how it shapes quantitative outcomes
    - Negative complementarities. $\frac{\sigma-1}{\theta} = \frac{2}{3}$
      Arkolakis, Ramondo, et al. 2018
    - Positive complementarities. $\frac{\sigma-1}{\theta} = \frac{3}{2}$

Outline

Details

Using the calibrated models, we:

1. illustrate the performance of our solution method, using some numerical exercises.
2. quantify the welfare gains from multinational production

Speed

Solving for the policy function (s)

	Negative Comp.			Positive Comp.		
Countries	Naive (1)	Sqz. (2)	Policy (3)	Naive (1)	Sqz. (2)	Policy (3)
8	8	0.423	0.019	7	0.480	0.034
16	5454	2.26	0.039	4356	2.36	0.087
32	–	11.1	0.11	–	13.2	0.19
64	–	66.0	1.32	–	94.5	1.29
128	–	456	14.1	–	795	14.7
256	–	3239	331	–	6479	374
Grid points	2^{14}	2^{14}	–	2^{14}	2^{14}	–

- Negative complementarities solve in comparable time
- Policy function is faster than incumbent (unhighlighted) approaches

The table reports the average time it takes to solve the firm's policy function, using three different solution methods.

- Naive: discretize productivity with $2^{14} \approx 16000$ grid points; then, at each grid point, compute profits at every combination of locations to find the profit-maximizing set
- Sqz: discretize as above, then apply squeezing and branching at each grid point
- Policy: apply the policy function approach

Highlighted columns are new approaches enabled by our method:

- Pos. comp.: Sqz. is the incumbent method pioneered by Antràs, Fort, and Tintelnot 2017
- Neg. comp.: previously no solution aside from brute force

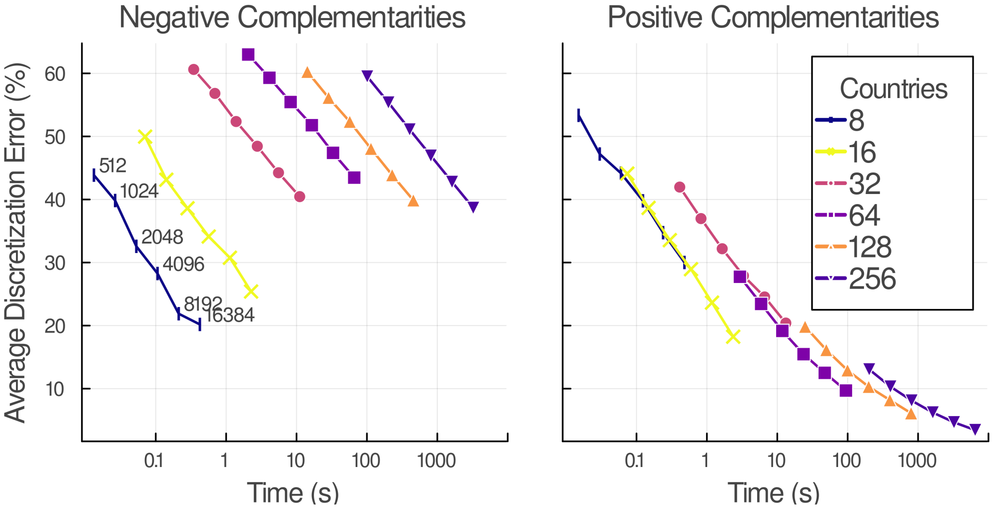
We vary the number of countries by generating "synthetic countries":

1. fit a distribution to calibrated model's estimated fundamentals

   e.g. bilateral trade costs, country labor productivity, etc

2. generate synthetic countries by sampling from distributions

Precision



Average percentage error in $X_{i\ell n}$ from discretization: drops 5–10p.p. each grid point doubling while policy function introduces no error

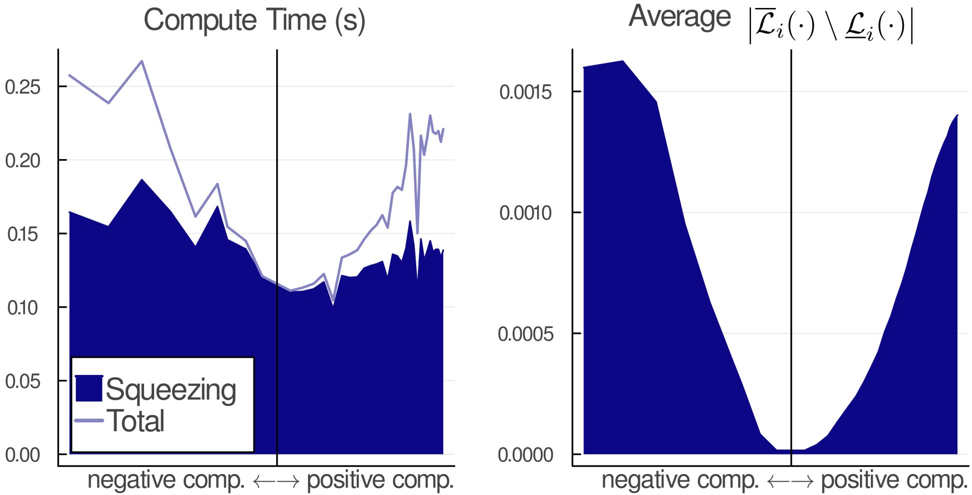
In addition to the speed disadvantage, methods with discretization introduce error since the policy function must be interpolated between grid points while the policy function is exact. Measuring discretization error:

- Use exact policy function to compute $X_{i\ell n}$: total sales from firms based in i to market n, produced in location ℓ
- Use discretized policy function to compute the approximation $\hat{X}_{i\ell n}$ by interpolating between grid points
- Average percentage error across triplets

$$\frac{1}{N^3}\sum_{i,\ell,n}\left|\frac{\hat{X}_{i\ell n}}{X_{i\ell n}}-1\right|\times 100\%$$

The plot traces the precision-time frontier by repeating this process, doubling the number of grid points each time. The nesting structure in positive complementarities case (Shannon and Milgrom 1994) reduces error.

Wide applicability



Fast computation across range of complementary (0.15–3.9); longer compute with strong complementarities

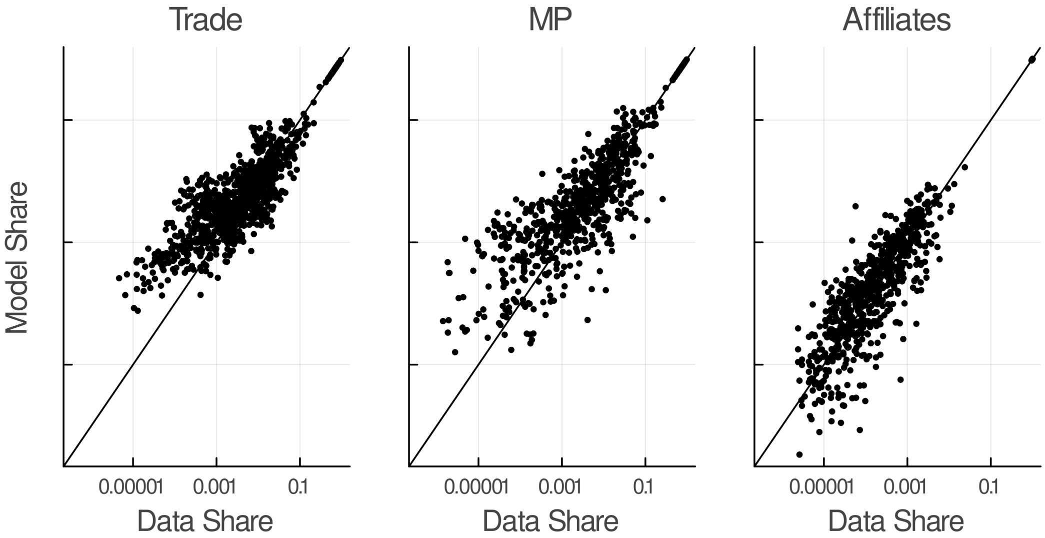
We vary the complementarity $\frac{\sigma-1}{\theta}$ from 0.15 to 3.9 by adjusting θ from 0.2 to 20, recalibrating the model for each value.

- The vertical line marks the knife-edge case of no complementarities ($\sigma - 1 = \theta$)
- The degree of complementarity on the horizontal axis is measured as $\frac{\sigma-1}{\theta} / \left(\frac{\sigma-1}{\theta} + 1\right)$

Both panels plot a measure of the generalized squeezing step's performance:

- Left panel:
  - Plots the average time to compute the firm's policy function in each corresponding calibrated model — fast even at high complementarity
  - Breaks out time taken by generalized squeezing, which does not vary largely by complementarity strength
- Right panel:
  - Plots the average number of locations in $\overline{\mathscr{L}}(z) \setminus \underline{\mathscr{L}}(z)$
  - Average number of leftover locations after generalized squeezing increases with stronger complementarities, consistent with longer compute time

Micro-data not required



Calibration matches aggregates shares (with negative complementarities)

Details

Click the plot to show the equivalent plot for the calibration with positive complementarities; click the positive complementarities plot to return back.

The speed of the policy function approach enables computing GE and calibrating to aggregate moments rather than firm micro-data.

Revisiting the welfare equation

- Arkolakis, Costinot, and Rodríguez-Clare 2012. The welfare change from reverting to autarky:

$$\ln \frac{\hat{w}_i}{\hat{P}_i} = \underbrace{\ln \hat{\pi}_{iii}^{-\frac{1}{\sigma-1}}}_{\text{openness}}$$

Details

We derive the counterfactual welfare change from returning to autarky. The formula applies whether we consider trade autarky or MP autarky (or both).

- $\pi_{i\ell n}$ is the share of consumption in n that is produced in ℓ by a firm headquartered in i
- $\tilde{z}_i$ is the productivity of the lowest-productivity firm headquartered in i (survival cutoff)
- $\mathscr{T}_i$ is the partitioning of the productivity range induced by $\mathscr{L}_i^{\star}(\cdot)$
- $s_{i\ell n}^t$ is the share of all sales produced in ℓ by firms in interval $\mathscr{Z}_i^t$: $\sum_\ell s_{i\ell n}^t = 1$ for all i, n, t
- $\lambda_{i\ell n}^t$ is the share of sales produced by firms in interval t: $\sum_t \lambda_{i\ell n}^t = 1$ for all i, ℓ, n

Revisiting the welfare equation

- Arkolakis, Costinot, and Rodríguez-Clare 2012. The welfare change from reverting to autarky, revisited:

$$\ln \frac{\hat{w}_i}{\hat{P}_i} = \underbrace{\ln \hat{\pi}_{iii}^{-\frac{1}{\sigma-1}}}_{\text{openness}} + \underbrace{\ln \hat{M}_i^{\frac{1}{\sigma-1}} + \ln \hat{\tilde{z}}_i^{-\frac{\zeta}{\sigma-1}}}_{\text{varieties}}$$

$$+ \underbrace{\ln \hat{\tilde{z}}_i + \ln \left[\sum_{\mathcal{Z}_i^t \in \mathcal{T}_i} \lambda_{iii}^t \left(s_{iii}^t \right)^{\frac{\sigma-1}{\theta}-1} \right]^{\frac{1}{\sigma-1}}}_{\text{average productivity}}$$

- General "openness". Applies for either trade and MP autarky

Welfare channels

Openness. Standard term captures reduction in real consumption, usually negative

$$\ln \hat{\pi}_{iii}^{-\frac{1}{\sigma-1}}$$

Varieties.

$$\ln \hat{M}_i^{\frac{1}{\sigma-1}} + \ln \hat{\tilde{z}}_i^{-\frac{\zeta}{\sigma-1}}$$

- Trade and MP autarky shrink the profits of previously large firms engaged in these foreign activities $\rightarrow$ Selection cutoff falls
- More entry and easier survival
- Usually: more varieties, positive effect

The sign of the variety and productivities channels depend on the degree to which countries are headquarters for MNEs compared to host countries for the foreign affiliates of MNEs headquartered abroad (as illustrated in quantification below).
We highlight the impact of complementarities on these channels:

- Varieties:
  - With positive complementarities, previously large "global" firms are larger than with negative complementarities
  - Variety effect tends to be more positive with positive than negative complementarities
- Productivity:
  - With positive complementarities:
    - the most productive firms shrink the most as they lose the scale economies that supported their large domestic market share
    - intensive margin effect is always negative
  - With negative complementarities:
    - the most productive firms expand relative to other firms in the domestic market, as they substitute foreign production with domestic production
    - itensive margin effect always positive

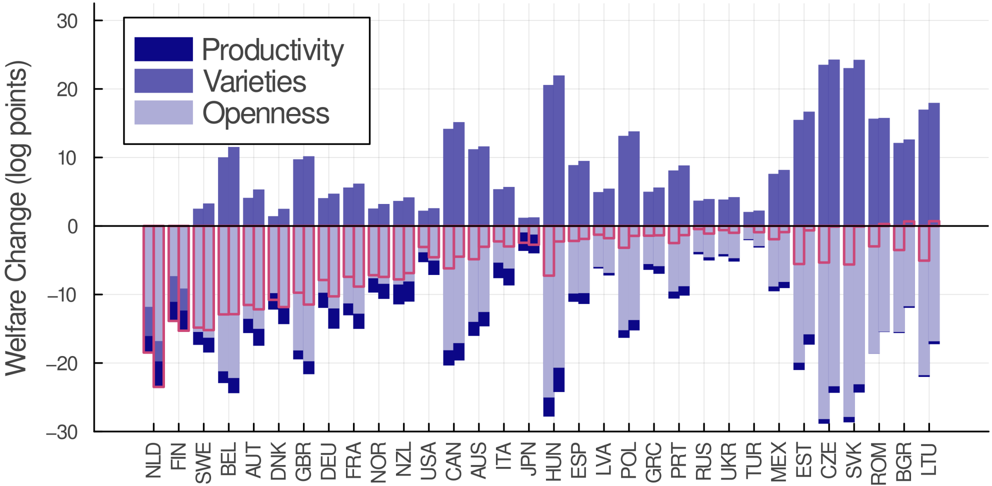
Welfare channels

Productivity.

$$\ln \hat{\tilde{z}}_i + \ln \left[\sum_{\mathcal{Z}_i^t \in \mathcal{T}_i} \lambda_{iii}^t \left(s_{iii}^t \right)^{\frac{\sigma-1}{\theta} - 1} \right]^{\frac{1}{\sigma-1}}$$

- Extensive margin: since selection cutoff falls, usually negative
- Intensive margin:
  - Trade and MP autarky shrink the profits of previously large firms engaged in these foreign activities $\rightarrow$ Relative firm sizes adjust
  - Sales-weighted in average productivity changes

MP autarky: Quantification



Left bars: results from the calibration with negative complementarities (right bars: positive complementarities)

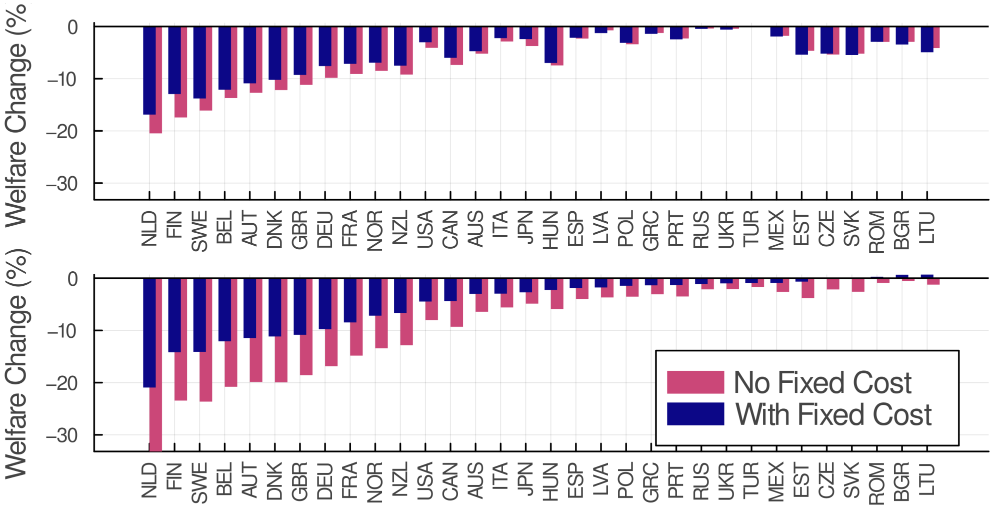
MP autarky: highlights the role of multinational location choices with complementarities and fixed costs

- Total welfare effect shown in pink
- Decomposition into three channels in blues

Most countries suffer negative welfare consequences from removing MP:

- Effect is less negative with negative compared to positive complementarities
- Productive, open economies suffer the most — access to MP is so valuable in these countries that entry *drops* in autarky for some
- Small, low-wage countries with few MNEs of their own lose the least
  - Lossess offset by positive variety effect
  - Departing foreign MNEs release labor that supports creating new domestic varieties

MP autarky: Quantification



Top figure: results from calibrations with negative complementarities (bottom: positive complementarities)

Details

Compares MP autarky counterfactual in benchmark calibrations against alternative calibrations without fixed costs.
Without fixed costs,

- Firms open production locations everywhere and $\ln \hat{\bar{z}}_i = 1$
- All affected symmetrically by MP autarky
- Welfare losses are larger compared to with fixed costs
  - With fixed costs, retreating MNEs free up labor that can instead create new domestic varieties, partially offsetting the welfare loss
  - Calibrations with no fixed cost overstate loss from MP autarky

To conclude

- Combinatorial discrete choice problems are common
  - Trade: multinational production, either export platforms or GVCs; firm sourcing partners; extended gravity export destinations
  - IO: input complementarity; product mix
  - (International) macro: tax avoidance and profit shifting, portfolio choice
  - Spatial economics: transport networks; real estate choices
  - ...
- We develop a new approach to CDCPs
  - With negative or positive complementarities
  - Policy function solution for aggregation
- Julia package: `CDCP.jl`

Part III

Appendix

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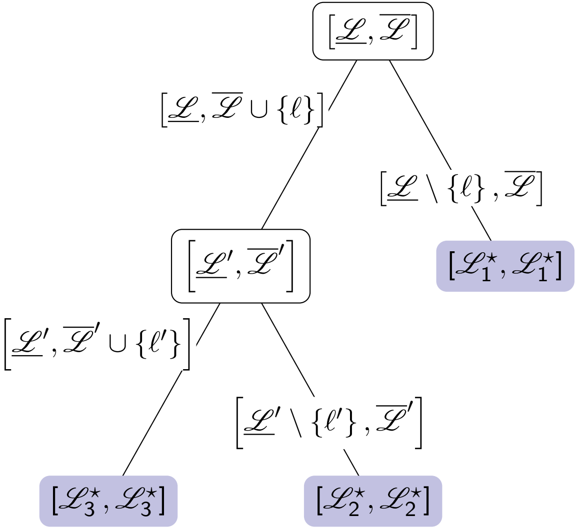
Branching tree

- convergence when bounding pair coincides on each branch
- branching collects all fixed points of Φ

invariant to the items selected and order

back



Cost minimization for each destination n

- the unit cost of producing an input υ at location $\ell \in \mathscr{L}$ then delivering it to market n is

$$\gamma_{i\ell} \frac{w_\ell}{z \varphi_\ell(\upsilon)} \tau_{\ell n}$$

$\gamma_{i\ell}$	arms-length iceberg cost of MP
w_ℓ	labor cost in production location
z	firm productivity
$\varphi_\ell(\upsilon)$	location-input shifter
$\tau_{\ell n}$	iceberg cost of trade

- tractable export platforms: for each destination n and input υ, the firm chooses the least-cost production location

Cost minimization for each destination n

- negative complementarity: marginal cost declines in $\mathscr{L}$, but decreasingly so (cannibalization)
  Antràs, Fort, and Tintelnot 2017; Tintelnot 2017

$$c_{in}(\mathscr{L}; z) = \left[\int_{\varphi} \left(\min_{\ell \in \mathscr{L}} \gamma_{i\ell} \frac{w_\ell}{z\varphi_\ell} \tau_{\ell n}\right)^{1-\eta} \mathrm{d}F(\varphi; \mathscr{L})\right]^{\frac{1}{1-\eta}}$$

- closed form with Fréchet location-input draws ($\eta < \theta + 1$)
  Arkolakis, Ramondo, et al. 2018; Ramondo and Rodríguez-Clare 2013; Lind and Ramondo 2023

$$c_{in}(\mathscr{L}; z) = \frac{1}{z}\Gamma \left[\sum_{\ell \in \mathscr{L}} \left(\frac{\gamma_{i\ell} w_\ell \tau_{\ell n}}{T_\ell}\right)^{-\theta}\right]^{-\frac{1}{\theta}}$$

back

General demand function

- sufficient condition for supermodularity

$$\underbrace{\varepsilon_D}_{\text{demand elasticity}} \times \underbrace{\frac{\mathrm{d}\ln p}{\mathrm{d}\ln c}}_{\text{passthrough}} \geq \underbrace{\theta}_{\text{cannibalization}} + 1$$

  - compares (positive) demand-side complementarity with (negative) supply-side complementarity
  - sufficient condition for submodularity: flip the sign
- flexible framework for discrete decisions and complementarities

back

Policy function: Japan

- with negative complementarities

```
[-Inf, 0.65]    +String[]
[0.65, 3.14]    +["JPN"]
[3.14, 3.302]   +["ROM"]
[3.302, 3.351]  +["ITA"]
[3.351, 3.403]  +["GBR"]  -["ITA", "ROM"]
[3.403, 3.574]  +["ITA"]
[3.574, 3.631]  +["ROM"]
...
```

- with positive complementarities

```
[-Inf, 0.666]   +String[]
[0.666, 4.253]  +["JPN"]
[4.253, 4.354]  +["DEU", "GBR", "FRA", "ITA", "POL", "ROM"]
...
```

back

Aggregate conditions

- Free entry. Require f_i^e to draw productivity

$$w_i f_i^e = \frac{1}{\sigma} \sum_n X_n \int \left(\frac{z P_n}{\mu} \right)^{\sigma - 1} \Theta_{in} \left(\mathscr{L}_i^\star(z) \right)^{\frac{\sigma - 1}{\theta}} \mathrm{d} G_i(z) - \int \sum_{\ell \in \mathscr{L}_i^\star(z)} w_\ell f_{i\ell} \mathrm{d} G_i(z)$$

- Price index. Aggregates over all firm origins i

$$P_n^{1-\sigma} = \sum_i M_i \int \left(\frac{z}{\mu} \right)^{\sigma - 1} \Theta_{in} \left(\mathscr{L}_i^\star(z) \right)^{\frac{\sigma - 1}{\theta}} \mathrm{d} G_i(z)$$

Aggregate conditions

- Labor market clearing. Inelastically supplied H_ℓ units

$$
\begin{aligned}
w_\ell H_\ell = \frac{\sigma-1}{\sigma} \sum_{i,n} X_n M_i \int & \frac{\mathbf{1}^\star_{i\ell}(z) \left(w_\ell \gamma_{i\ell} \tau_{\ell n} / T_\ell\right)^{-\theta}}{\sum_{k \in \mathscr{L}^\star_i(z)} \left(w_k \gamma_{ik} \tau_{kn} / T_k\right)^{-\theta}} \\
& \times \left(\frac{z P_n}{\mu}\right)^{\sigma-1} \Theta_{in}\left(\mathscr{L}^\star_i(z)\right)^{\frac{\sigma-1}{\theta}} \mathrm{d}G_i(z) \\
& + \sum_i M_i \int \mathbf{1}^\star_{i\ell}(z) w_\ell f_{i\ell} \mathrm{d}G_i(z) + M_\ell w_\ell f^e_\ell
\end{aligned}
$$

- Balance of payments.

$$
X_n = w_n H_n
$$

back

Quantification

- Parameterization.
  - $g_i(\cdot) \sim$ Pareto with shape ζ and minimum $\underline{z}_i$
  - bilateral trade, MP, and fixed costs with gravity variables
- Calibration strategy.

Parameter	Target
σ	set to 4
	Arkolakis, Ramondo, et al. 2018; Head and Mayer 2019
ζ	firm sales tail
	Arkolakis 2010
$T_\ell, \underline{z}_i$	GDP, total foreign MP outgoing
f_i, f_i^e	enterprise survival rate, count
$\tau_{\ell n}, \gamma_{i\ell}, \nu_{i\ell}$	trade, MP, and affiliate flow
	details

back

Bilateral costs

- Parameterization. Gravity variables $v \in \{\log \text{dist}, \text{COL}, \text{BOR}, \text{COM}\}$
  Conte, Cotterlaz, Mayer, et al. 2023

$$\log \tau_{\ell n} = \sum_v \kappa_\tau^v v_{\ell n} + \mathbf{1}[\ell \neq n]\overline{\tau}_n + \log(1 + t_{\ell n})$$

$$\log \gamma_{i\ell} = \sum_v \kappa_\gamma^v v_{i\ell} + \mathbf{1}[i \neq \ell]\overline{\gamma}_n$$

$$\log f_{i\ell} = \sum_v \kappa_f^v v_{i\ell} + \mathbf{1}[i \neq \ell]\overline{f}_n$$

- Match aggregate flows.
  - $\kappa_\tau^v, \kappa_\gamma^v, \kappa_f^v$ corresponding coefficient on gravity variables in trade, MP, and affiliate regressions
  - $\overline{\tau}_n, \overline{\gamma}_\ell, \overline{f}_\ell$ own shares of "trade" (absorption), "MP" (domestic production), and affiliates (domestic production locations)

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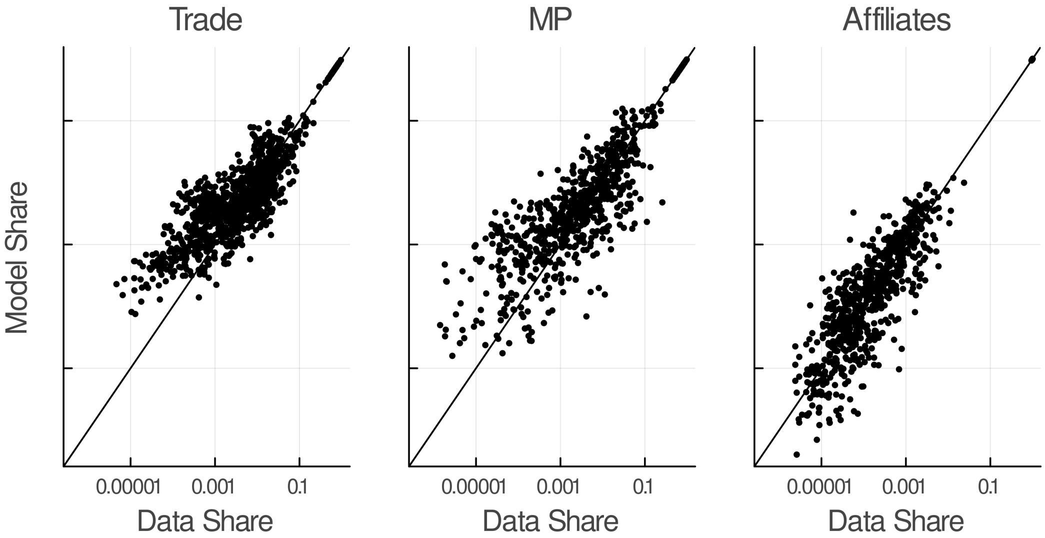
Micro-data not required



Calibration matches aggregates shares (with positive complementarities)