Combinatorial Discrete Choice

Teaching slides

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Version 0.1

Motivation

- Discrete choice problems with complementarities among options
  - Tesla choosing in which countries to operate production plants
  - Starbucks choosing blocks in Manhattan to operate shops
  - A government choosing locations for critical infrastructure
- Without more structure: an intractable NP hard problem
- This paper. Solve such combinatorial discrete choice problems
- Key. Economic complementarities provide exploitable structure

Part I

Theory

Notation

- Set of discrete options L
  Index individual items in L by ℓ, so that $\ell \in L$
- Define collection of subsets (power set) of L as: $\mathscr{P}(L)$
  Denote individual element in $\mathscr{P}(L)$ by $\mathscr{L}$, so that $\mathscr{L} \in \mathscr{P}(L)$
- Define the space of objective functions $\mathscr{F} = \{f : \mathscr{P}(L) \to \mathbb{R}\}$
  Denote an individual objective by f, so that $f \in \mathscr{F}$

Outline

Characterization

- Maximization over subsets. Choose the subset of items $\mathscr{L} \subseteq L$

  leading example: multinational location problem

$$\mathscr{L}^\star = \arg\max_{\mathscr{L} \subseteq L} f(\mathscr{L})$$

- Marginal value operator. For an item ℓ, the value with it compared to without it, contingent on $\mathscr{L}$

  discrete analogue to derivative

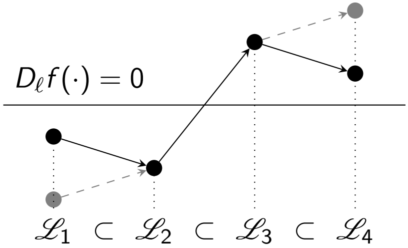
$$D_\ell f(\mathscr{L}) = f(\mathscr{L} \cup \{\ell\}) - f(\mathscr{L} \setminus \{\ell\})$$

- *Combinatorial* discrete choice. If the marginal value varies with $\mathscr{L}$

Single crossing differences in choices

From below. If ℓ is valuable given a small set, *remains* valuable given a large set:

$$D_\ell f\left(\mathscr{L}\right) \geq 0 \quad \Rightarrow \quad D_\ell f\left(\mathscr{L}'\right) \geq 0$$



Supermodularity. *More* valuable given large set compared to small set

$$D_\ell f\left(\mathscr{L}\right) \leq D_\ell f\left(\mathscr{L}'\right)$$

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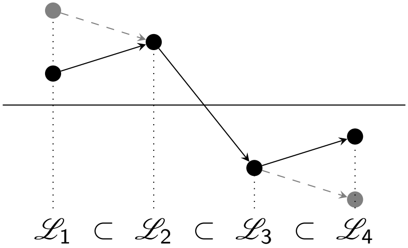


Submodularity. *More* valuable given small set compared to large set

$$D_\ell f\left(\mathscr{L}\right) \geq D_\ell f\left(\mathscr{L}'\right)$$

Single crossing differences in choices

Definition (Quasi-supermodularity and quasi-submodularity)

The function f is:

a) *quasi-supermodular if, for all $\mathscr{L}, \mathscr{L}' \in \mathscr{P}(L)$,*

$$f\left(\mathscr{L} \cup \mathscr{L}'\right) \leq f\left(\mathscr{L}'\right) \qquad \Rightarrow \qquad f\left(\mathscr{L}\right) \leq f\left(\mathscr{L} \cap \mathscr{L}'\right)$$

b) *quasi-submodular if, for all $\mathscr{L}, \mathscr{L}' \in \mathscr{P}(L)$,*

$$f\left(\mathscr{L}\right) \geq f\left(\mathscr{L} \cap \mathscr{L}'\right) \qquad \Rightarrow \qquad f\left(\mathscr{L} \cup \mathscr{L}'\right) \geq f\left(\mathscr{L}'\right)$$

Shannon and Milgrom 1994; Milgrom 2004

Corollary

Quasi-supermodularity is sufficient for SCD-C from below; quasi-submodularity is sufficient for SCD-C from above.

"Local optimality"

- Jia 2008. Central mapping:

$$\Phi(\mathscr{L}) = \{\ell \in L \mid D_\ell f(\mathscr{L}) \geq 0\}$$

"All items with non-negative marginal value to $\mathscr{L}$"

- No deviation by one element. Necessary, not sufficient!

similar to a first order condition

$$\mathscr{L}^\star = \Phi(\mathscr{L}^\star)$$

  - if ℓ is chosen ($\ell \in \mathscr{L}^\star$), then it must contribute positive marginal value ($\ell \in \Phi(\mathscr{L}^\star)$)
  - if ℓ is *not* chosen ($\ell \notin \mathscr{L}^\star$), then it cannot add value when included ($\ell \notin \Phi(\mathscr{L}^\star)$)

Order-preserving (reversing)

Lemma

If f satisfies SCD-C from below (above), Φ is order-preserving (reversing).

Squeezing mapping

- Bounding pair $[\underline{\mathscr{L}}, \overline{\mathscr{L}}]$. Defines a restricted domain

$$\{\mathscr{L} | \underline{\mathscr{L}} \subseteq \mathscr{L} \subseteq \overline{\mathscr{L}}\} \subseteq \mathscr{P}(L)$$

  - the full domain is represented $[\emptyset, L] = \mathscr{P}(L)$
  - $[\underline{\mathscr{K}}, \overline{\mathscr{K}}]$ is "tighter" than $[\underline{\mathscr{L}}, \overline{\mathscr{L}}]$ if $[\underline{\mathscr{K}}, \overline{\mathscr{K}}] \subseteq [\underline{\mathscr{L}}, \overline{\mathscr{L}}]$, i.e. it defines a subdomain
- Squeezing mapping. Acts on bounding pairs

$$S\left([\underline{\mathscr{L}}, \overline{\mathscr{L}}]\right) = \left[\inf\left\{\Phi\left(\underline{\mathscr{L}}\right), \Phi\left(\overline{\mathscr{L}}\right)\right\}, \sup\left\{\Phi\left(\underline{\mathscr{L}}\right), \Phi\left(\overline{\mathscr{L}}\right)\right\}\right]$$

- Iterative application. Let $S^k\left([\underline{\mathscr{L}}, \overline{\mathscr{L}}]\right)$ denote applying S iteratively k times

Main theorem: Single agent problem

Theorem 1 (Squeezing procedure)

If f satisfies SCD-C, then:

a. *let* $\left[\underline{\mathscr{L}}^{(k)}, \overline{\mathscr{L}}^{(k)}\right] \equiv S^k([\emptyset, L])$; *then,*

$$\emptyset \subseteq \ldots \subseteq \underline{\mathscr{L}}^{(k)} \subseteq \underline{\mathscr{L}}^{(k+1)} \subseteq \overline{\mathscr{L}}^{(k+1)} \subseteq \overline{\mathscr{L}}^{(k)} \subseteq \ldots \subseteq L$$

"iterative application weakly tightens the problem's domain"

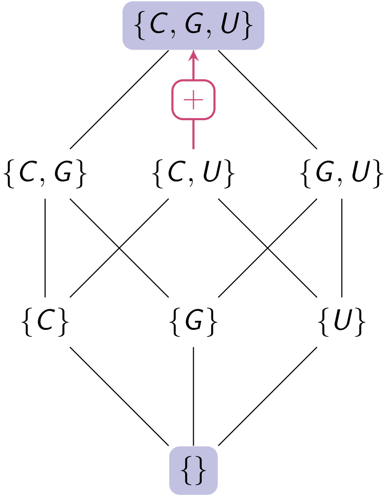
b. *if* $\mathscr{L}^\star \in [\mathscr{L}, \mathscr{L}']$, *then* $\mathscr{L}^\star \in S([\mathscr{L}, \mathscr{L}'])$

"if the optimum set is in the restricted domain, S will not discard it"

c. $S^{|L|}([\emptyset, L]) = S^{|L|+1}([\emptyset, L])$

"iterating the squeezing step S converges to a fixed point in $|L|$ steps or fewer"

Proof intuition: SCD-C from above



- Bounding pair. $\underline{\mathscr{L}} \subseteq \mathscr{L}^\star \subseteq \overline{\mathscr{L}}$

  $\underline{\mathscr{L}}$ tracks elements in $\mathscr{L}^\star$

  $\overline{\mathscr{L}}$ discards elements not in $\mathscr{L}^\star$

  $\overline{\mathscr{L}} \setminus \underline{\mathscr{L}}$ tracks elements maybe in $\mathscr{L}^\star$
- Rule out suboptimal strategies.
  - check marginal value at points of extreme complementarity
  - iteratively squeeze: update the subset and superset

    halve decision space each time

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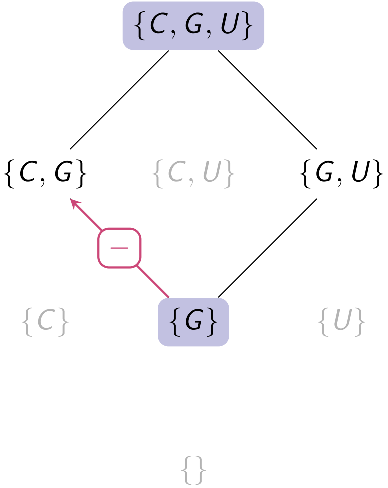

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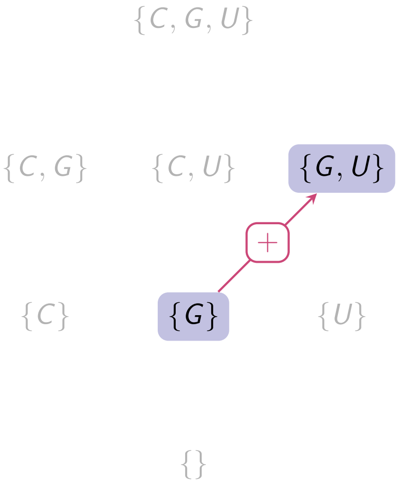

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$\{C, G, U\}$

$\{C, G\}$ $\{C, U\}$ $\{G, U\}$

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$\{\}$

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- **Rule out suboptimal strategies.**
  - check marginal value at points of extreme complementarity
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    halve decision space each time

SCD-C from below

Lattice foundation

- Jia 2008. Solution method for supermodular f:
  1. Central mapping. By construction, $\mathscr{L}^\star$ is a fixed point of:

  $$\Phi\left(\mathscr{L}\right) \equiv \left\{\ell \in L \mid D_\ell f\left(\mathscr{L}\right) \geq 0\right\}$$

  2. Order-preserving Φ. With supermodular f
  3. Tarski 1955. Order-preserving Φ has a smallest and largest fixed point ...
  4. Kleene 1936. ...identified by iterating $\Phi^\infty\left(\emptyset\right)$ and $\Phi^\infty\left(L\right)$
- SCD-C (from below). Necessary and sufficient condition for Φ to be order-preserving

SCD-C from above

Lattice foundation

- Order-reversing Φ. Tarski 1955; Kleene 1936 no longer apply
- Perfect substitutes intuition. Consider two elements, $\{a, b\}$
  - both items have positive marginal value in isolation, but neither have positive marginal value if the other is included

$$\Phi(\emptyset) = \{a, b\} \qquad \Phi(\{a, b\}) = \emptyset$$

  - the fixed points are uncomparable, i.e. there is neither a smallest nor largest fixed point — Tarski 1955 breaks down ...

$$\Phi(\{a\}) = \{a\} \qquad \Phi(\{b\}) = \{b\}$$

  - ...without the existence of smallest and largest fixed points, does iteration converge? To what?

SCD-C from above

Lattice foundation

A generalization of the notion of a fixed point:

Definition (Fixed edge)

Two sets, $\mathscr{L}$ and $\mathscr{L}'$ with

$$\Phi\left(\mathscr{L}\right) = \mathscr{L}' \qquad , \qquad \Phi\left(\mathscr{L}'\right) = \mathscr{L}$$

- **Klimeš 1981.** Order-reversing Φ has an "extreme" fixed edge $\mathscr{L}^{\text{inf}}, \mathscr{L}^{\text{sup}}$!

$$\mathscr{L}^{\text{inf}} \subseteq \mathscr{L} \subseteq \mathscr{L}' \subseteq \mathscr{L}^{\text{sup}}$$

- **Iteration.** $\lim_{n\to\inf} \Phi^{2n}\left(\emptyset\right) = \mathscr{L}^{\text{inf}}$ and $\lim_{n\to\inf} \Phi^{2n+1}\left(\emptyset\right) = \mathscr{L}^{\text{sup}}$
  vice versa from L

SCD-C from above

Lattice foundation

- Φ's "Fixed edge convergence". After enough applications, the mapping Φ alternates back and forth between the two points in the fixed edge
- Squeezing step. Converges to fixed point by construction:

$$S\left(\left[\mathscr{L}^{\text{inf}}, \mathscr{L}^{\text{sup}}\right]\right) = \left[\Phi\left(\mathscr{L}^{\text{sup}}\right), \Phi\left(\mathscr{L}^{\text{inf}}\right)\right] = \left[\mathscr{L}^{\text{inf}}, \mathscr{L}^{\text{sup}}\right]$$

by "flipping" the order of the two sets

Refinement: branching

- If $\mathscr{L}^{\text{inf}} = \mathscr{L}^{\text{sup}}$, then $\mathscr{L}^{\text{inf}} = \mathscr{L}^{\star}$
- Sometimes: converge, but $\mathscr{L}^{\text{inf}} \subset \mathscr{L}^{\star}$

  e.g. when complementarities very strong

$\{C, G, U\}$

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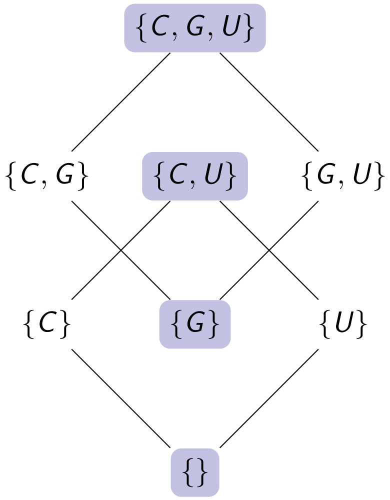
- Choose an item $\ell \in \overline{\mathscr{L}} \setminus \underline{\mathscr{L}}$, then
  - divide into two subproblems: with and without ℓ
  - squeeze on each problem, branching as needed tree



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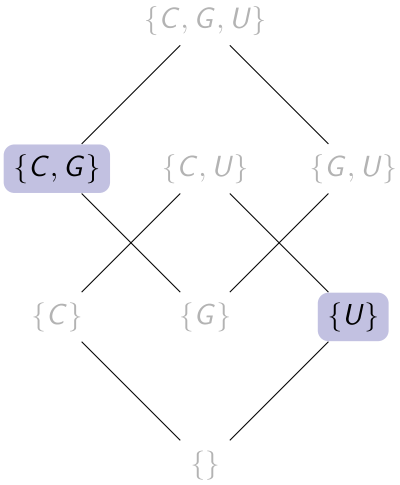
- Choose an item $\ell \in \overline{\mathscr{L}} \setminus \underline{\mathscr{L}}$, then
  - divide into two subproblems: with and without ℓ
  - squeeze on each problem, branching as needed tree
- End: "conditionally optimal" decision sets
  - among them, the global optimum
  - intuition: "brute force" one decision at a time, squeeze as much as possible



Outline

Heterogeneous agent problem

- Augmented objective function. $f : \mathscr{P}(L) \times \mathbb{R} \to \mathbb{R}$ maps the set $\mathscr{L}$ and the agent type $z \in \mathbb{R}$ to a scalar payoff $f(\mathscr{L}, z)$
  leading example: multinational location problem with heterogeneous productivity
- Policy function. Function $\mathscr{L}^\star : \mathbb{R} \to \mathscr{P}(L)$ specifies the optimal decision set for each type z:

$$\mathscr{L}^\star(z) = \arg\max_{\mathscr{L} \in \mathscr{P}(L)} f(\mathscr{L}, z)$$

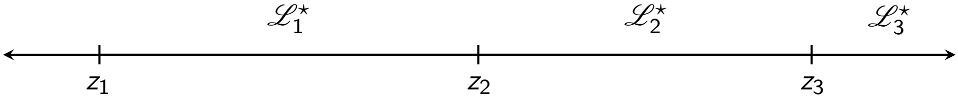
Single crossing differences in types

- **SCD in types (SCD-T).** For all elements $\ell \in L$, decision sets $\mathscr{L} \in \mathscr{P}(L)$, and types $z, z' \in \mathbb{R}$ such that $z < z'$,

$$D_\ell f(\mathscr{L}, z) \geq 0 \qquad \Rightarrow \qquad D_\ell f(\mathscr{L}, z') \geq 0$$

  SCD-T is equivalent to the single-crossing differences condition of Milgrom 2004 (originally "single crossing" in Shannon and Milgrom 1994).

- **With SCD-C and SCD-T.** The policy function changes its value only at a finite number of cutoff productivities:



- **Approach.** Partition type space into intervals that share the same policy; and find policy associated with each interval

Type space partition

- Bounding set functions. Extend bounding pair to set-valued functions $\underline{\mathscr{L}}(\cdot)$ and $\overline{\mathscr{L}}(\cdot)$ with

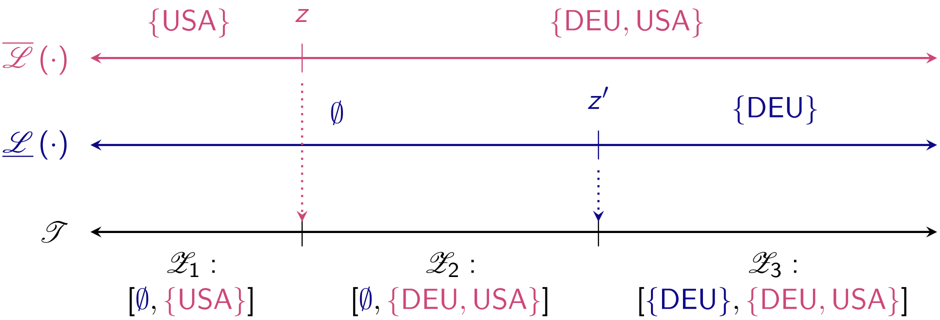
$$\underline{\mathscr{L}}(z) \subseteq \mathscr{L}^{\star}(z) \subseteq \overline{\mathscr{L}}(z)$$

  for any productivity $z \in \mathbb{R}$
  trivial bounding set functions: constant functions $[\emptyset, L]$

- Induced partition. From bounding set functions:

$$\mathscr{T}\left(\left[\underline{\mathscr{L}}(\cdot), \overline{\mathscr{L}}(\cdot)\right]\right) = \{\mathscr{Z}_1, \ldots \mathscr{Z}_t, \ldots \mathscr{Z}_T\}$$
$$\text{such that } \mathscr{Z}_t = \left\{z \in \mathbb{R} \mid \underline{\mathscr{L}}(z) = \underline{\mathscr{L}}_t, \overline{\mathscr{L}}(z) = \overline{\mathscr{L}}_t\right\},$$

Type space partition



Together, the two set-valued functions imply the partitioning $\mathscr{T}$, which creates intervals of productivity.

Identifying cutoffs: intuition

- SCD-C. “Choice monotonicity” rules out decision sets without explicitly evaluating their payoff; together with . . .
- SCD-T. “Type monotonicity” means choice monotonicity can discard decision sets for productivity ranges without evaluating the objective at any of the productivities

Generalized squeezing

- With SCD-C + SCD-T. For each ℓ and $\mathscr{L}$, there is s unique type indifferent between including ℓ in $\mathscr{L}$

$$0 = D_\ell \left(\mathscr{L}, z_\ell^g\left(\mathscr{L}\right)\right)$$

- Indifferent type. Use to avoid evaluating $\Phi(\mathscr{L}, z)$ at each z for a given $\mathscr{L}$:

$$\Phi^g\left(\mathscr{L}, z\right) = \left\{\ell \mid z \geq z_\ell^g\left(\mathscr{L}\right)\right\}$$

- Generalized squeezing mapping.

$$\begin{aligned} S^g\left(\left[\mathscr{L}\left(\cdot\right), \mathscr{L}'\left(\cdot\right)\right]\right) = \Big[&\inf\left\{\Phi^g\left(\mathscr{L}\left(\cdot\right), \cdot\right), \Phi^g\left(\mathscr{L}'\left(\cdot\right), \cdot\right)\right\}, \\ &\sup\left\{\Phi^g\left(\mathscr{L}\left(\cdot\right), \cdot\right), \Phi^g\left(\mathscr{L}'\left(\cdot\right), \cdot\right)\right\}\Big] \end{aligned}$$

Main theorem: Policy function

Theorem 2 (Generalized squeezing procedure)

If f satisfies SCD-C and SCD-T,

a. *Theorem 1a. and 1b. hold at each z*

b. $(S^g)^{|L|}([\emptyset, L]) = (S^g)^{|L|+1}([\emptyset, L])$

Proof.

Let $\Phi(\mathscr{L}, z) \equiv \{\ell \mid D_\ell f(\mathscr{L}, z) \geq 0\}$ be the mapping Φ evaluated at the type z. Applying Theorem 1 element-wise, we have for all z that

$$\underline{\mathscr{L}}_t \subseteq \Phi\left(\overline{\mathscr{L}}_t, z\right) \subseteq \mathscr{L}^\star(z) \subseteq \Phi\left(\underline{\mathscr{L}}_t, z\right) \subseteq \overline{\mathscr{L}}_t \,.$$

Then, it suffices to show that, for all z, $\Phi^g(\mathscr{L}, z)$ coincides with $\Phi(\mathscr{L}, z)$. The proof uses SCD-C and SCD-T to establish this equivalence. □

Proof intuition: SCD-C from above

For a given interval $\mathcal{Z}_t \in \mathcal{T}$:

1. select $\ell \in \overline{\mathscr{L}}_t \setminus \overline{\mathscr{L}}_t$, compute the two cutoffs

$$z_\ell^g \left(\underline{\mathscr{L}}_t\right) \qquad \leq \qquad z_\ell^g \left(\overline{\mathscr{L}}_t\right)$$

2. update bounding set functions:
   - for all firms with productivity $z < z_\ell^g \left(\underline{\mathscr{L}}_t\right)$ in $\mathscr{Z}_t$, ℓ is not part of the optimal decision set: update upper bounding set function to $\overline{\mathscr{L}}_t \setminus \{\ell\}$ for these productivities
   - for all firms with productivity $z > z_\ell^g \left(\overline{\mathscr{L}}_t\right)$ in $\mathscr{Z}_t$, ℓ is in the optimal decision set: update lower bounding set function to $\underline{\mathscr{L}}_t \cup \{\ell\}$ for these productivities
3. repeat for all $\ell \in \overline{\mathscr{L}}_t \setminus \overline{\mathscr{L}}_t$
4. use new bounding set functions to update partition

Part II

Application: MNEs

Outline

Quantitative framework

- Multinational firm CDCP
- SCD-C and SCD-T in firm problem
- Policy function and aggregation

Solution at work

- Solution method's performance
- Quantitative counterfactual

A model of multinational activity

- Setup.
  - Firms are born in origin country with productivity $z \sim g(z)$
  - Each firm produces a differentiated variety, compete monopolistically
  - There are L potential production locations
- Firm problem overview.
  - CDCP. Firms choose production locations subject to complementarities among locations and fixed costs
  - Heterogeneity. Productivity differences $\rightarrow$ Firms choose different production location sets $\rightarrow$ MNEs arise endogenously
- Full GE. Endogenous wages, firm entry, ...

The firm problem

1. Location choice (extensive margin). Choose a set of production locations $\mathscr{L}$

   index origin country with i, production location with ℓ, destination market with n

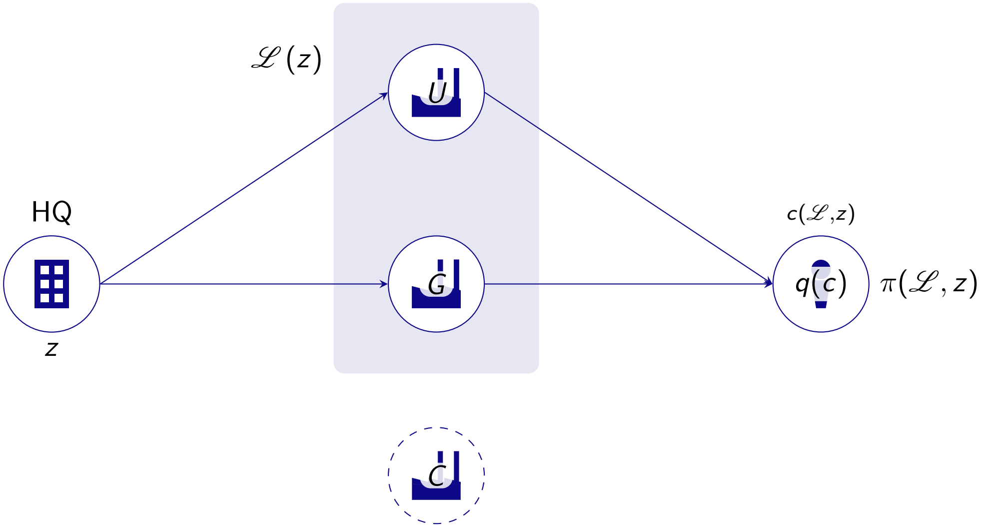
2. Price and quantity (intensive margin). Choose price (quantity), contingent on CES marginal cost

   a possible microfoundation: input sourcing (Tintelnot 2017; Antràs, Fort, and Tintelnot 2017; Arkolakis, Ramondo, et al. 2018) details

$$c_{in}\left(\mathscr{L}, z\right) = \frac{1}{z}\left[\sum_{\ell \in \mathscr{L}} \left(\frac{w_\ell \gamma_{i\ell} \tau_{\ell n}}{T_\ell}\right)^{-\theta}\right]^{-\frac{1}{\theta}} = \frac{1}{z}\left[\sum_{\ell \in \mathscr{L}} \xi_{i\ell n}^{-\theta}\right]^{-\frac{1}{\theta}}$$

- marginal cost declines in $\mathscr{L}$
- θ: substitutability (complementarity) among locations in cost

The firm problem



The firm problem

- CES demand. In each market n,
  - the firm sets constant markup $\mu = \frac{\sigma}{\sigma-1}$ over marginal cost
  - let X_n be total expenditure and P_n be CES price index
- Total profits. Adding up over destination markets:

$$\pi_i(\mathscr{L}, z) \equiv \left[1 - \frac{1}{\mu}\right] \sum_n X_n \left(\frac{zP_n}{\mu}\right)^{\sigma-1} \left[\sum_{\ell \in \mathscr{L}} \xi_{i\ell n}^{-\theta}\right]^{\frac{\sigma-1}{\theta}} - \sum_{\ell \in \mathscr{L}} w_\ell f_{i\ell}$$

  where $f_{i\ell}$ is the fixed labor cost of establishing production in location ℓ
- Location choice policy function. The firm chooses production locations to maximize total profits:

$$\mathscr{L}_i^\star(z) = \arg\max_{\mathscr{L} \subseteq L} \pi_i(\mathscr{L}, z)$$

Firm location choice is a CDCP

- Marginal value of location k. Trades off the marginal cost savings with fixed cost:

$$\frac{1}{\sigma}\sum_n X_n\left(\frac{zP_n}{\mu}\right)^{\sigma-1}\left\{\left[\xi_{ikn}^{-\theta}+\sum_{\mathscr{L}}\xi_{i\ell n}^{-\theta}\right]^{\frac{\sigma-1}{\theta}}-\left[\sum_{\mathscr{L}}\xi_{i\ell n}^{-\theta}\right]^{\frac{\sigma-1}{\theta}}\right\}-w_k f_{ik}$$

  complementarities preclude deciding on each location independently from the other locations

- Applying our solution. Establish SCD first:
  - SCD-C. Sufficient condition: $\sigma-1\lesseqgtr\theta$

    θ cost-side cannibalization (or complementarity)

    σ demand-side market-level scale effect

  - SCD-T. Sufficient condition: $\sigma>1$

general demand

Policy function in practice

- Policy function $\mathscr{L}_i(z)$. Maps firm productivity z to production location set example
- Aggregation. Production in location ℓ of the average active firm from origin i requires integrating over optimal decision set for each active type z

$$\sum_n X_n \left(\frac{zP_n}{\mu}\right)^{\sigma-1} \int \frac{\mathbf{1}^\star_\ell(z)\xi_{i\ell n}^{-\theta}}{\sum_{k\in\mathscr{L}^\star(z)} \xi_{ikn}^{-\theta}} \left[\sum_{\mathscr{L}_i^\star(z)} \xi_{i\ell n}^{-\theta}\right]^{\frac{\sigma-1}{\theta}} \mathrm{d}G_i\left(z \mid \text{active}\right)$$

- Gravity at the firm level (across locations), but not in the aggregate

Closing and quantifying the model

- Aggregate conditions. details
  - Free entry with entry labor cost f_i^e
  - Labor market clearing with H_ℓ units, inelastically supplied
  - Balance of payments
- Quantification. Calibrate with:
  - 32 countries using aggregate data details
    Alviarez 2019; Feenstra, Inklaar, and Timmer 2015
  - two levels of complementarities to highlight how it shapes quantitative outcomes
    - Negative complementarities. $\frac{\sigma-1}{\theta} = \frac{2}{3}$
      Arkolakis, Ramondo, et al. 2018
    - Positive complementarities. $\frac{\sigma-1}{\theta} = \frac{3}{2}$

Outline

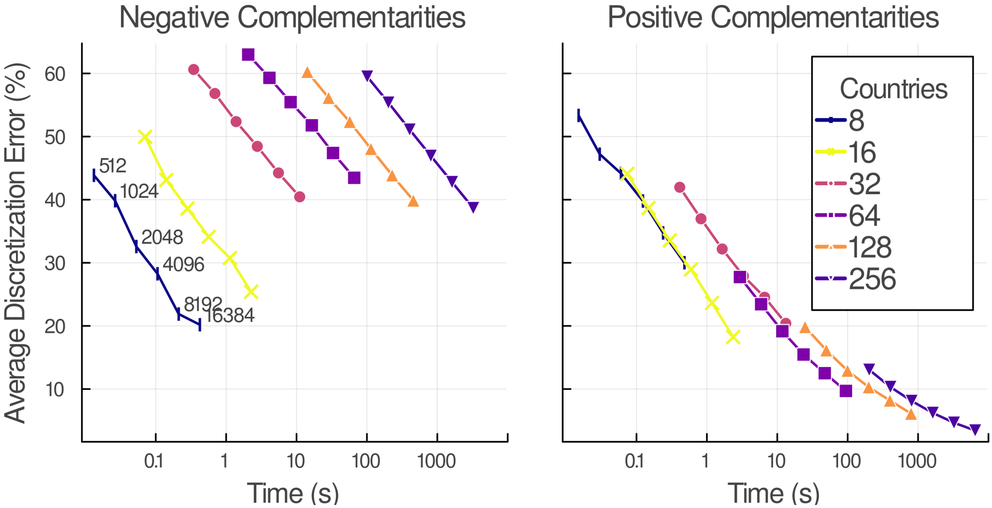
Speed

Solving for the policy function (s)

Countries	Negative Comp. Naive (1)	Negative Comp. Sqz. (2)	Negative Comp. Policy (3)	Positive Comp. Naive (1)	Positive Comp. Sqz. (2)	Positive Comp. Policy (3)
8	8	0.423	0.019	7	0.480	0.034
16	5454	2.26	0.039	4356	2.36	0.087
32	–	11.1	0.11	–	13.2	0.19
64	–	66.0	1.32	–	94.5	1.29
128	–	456	14.1	–	795	14.7
256	–	3239	331	–	6479	374
Grid points	2^{14}	2^{14}	–	2^{14}	2^{14}	–

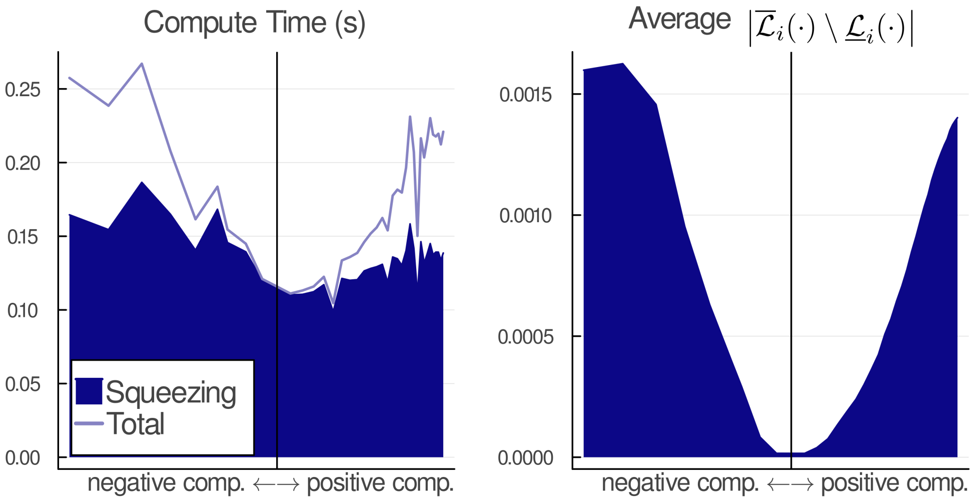
- Negative complementarities solve in comparable time
- Policy function is faster than incumbent (unhighlighted) approaches

Precision



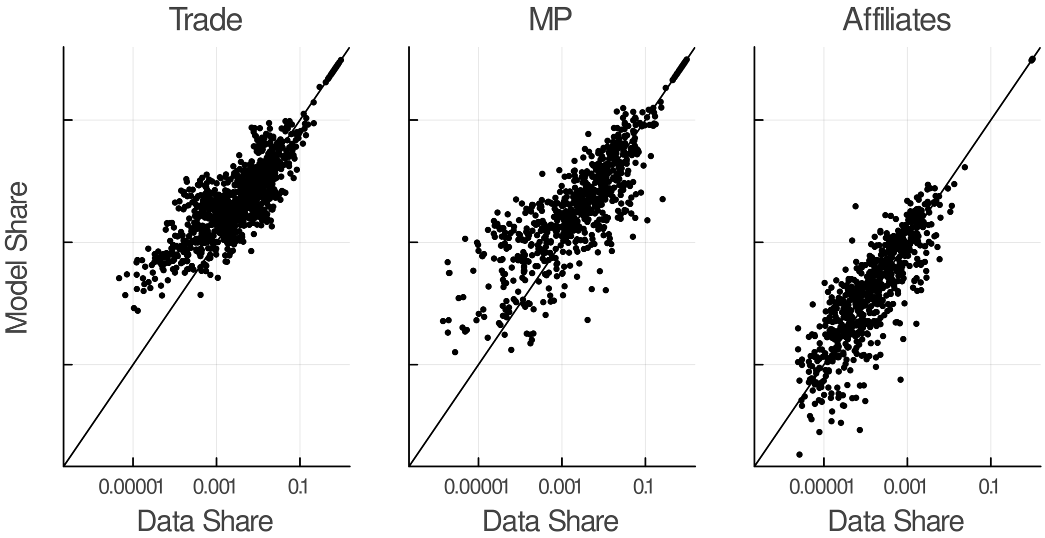
Average percentage error in $X_{i\ell n}$ from discretization: drops 5–10p.p. each grid point doubling while policy function introduces no error

Wide applicability



Fast computation across range of complementary (0.15–3.9); longer compute with strong complementarities

Micro-data not required



Calibration matches aggregates shares (with negative complementarities)

Revisiting the welfare equation

- Arkolakis, Costinot, and Rodríguez-Clare 2012. The welfare change from reverting to autarky:

$$\ln \frac{\hat{w}_i}{\hat{P}_i} = \underbrace{\ln \hat{\pi}_{iii}^{-\frac{1}{\sigma-1}}}_{\text{openness}}$$

Revisiting the welfare equation

- Arkolakis, Costinot, and Rodríguez-Clare 2012. The welfare change from reverting to autarky, revisited:

$$\ln \frac{\hat{w}_i}{\hat{P}_i} = \underbrace{\ln \hat{\pi}_{iii}^{-\frac{1}{\sigma-1}}}_{\text{openness}} + \underbrace{\ln \hat{M}_i^{\frac{1}{\sigma-1}} + \ln \hat{\tilde{z}}_i^{-\frac{\zeta}{\sigma-1}}}_{\text{varieties}}$$

$$+ \underbrace{\ln \hat{\tilde{z}}_i + \ln \left[\sum_{\mathcal{Z}_i^t \in \mathcal{T}_i} \lambda_{iii}^t \left(s_{iii}^t \right)^{\frac{\sigma-1}{\theta}-1} \right]^{\frac{1}{\sigma-1}}}_{\text{average productivity}}$$

- General "openness". Applies for either trade and MP autarky

Welfare channels

Openness. Standard term captures reduction in real consumption, usually negative

$$\ln \hat{\pi}_{iii}^{-\frac{1}{\sigma-1}}$$

Varieties.

$$\ln \hat{M}_i^{\frac{1}{\sigma-1}} + \ln \hat{\tilde{z}}_i^{-\frac{\zeta}{\sigma-1}}$$

- Trade and MP autarky shrink the profits of previously large firms engaged in these foreign activities $\rightarrow$ Selection cutoff falls
- More entry and easier survival
- Usually: more varieties, positive effect

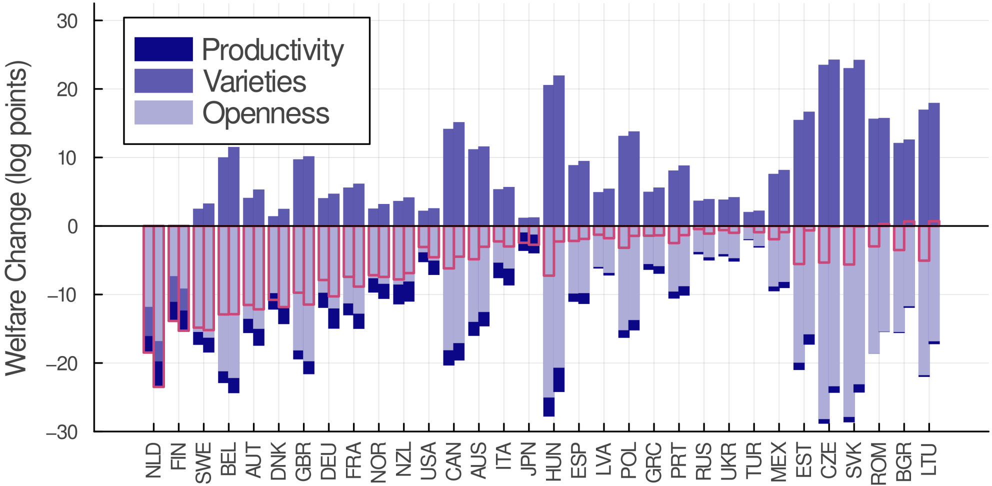
Welfare channels

Productivity.

$$\ln \hat{\tilde{z}}_i + \ln \left[\sum_{\mathcal{Z}_i^t \in \mathcal{T}_i} \lambda_{iii}^t \left(s_{iii}^t \right)^{\frac{\sigma-1}{\theta}-1} \right]^{\frac{1}{\sigma-1}}$$

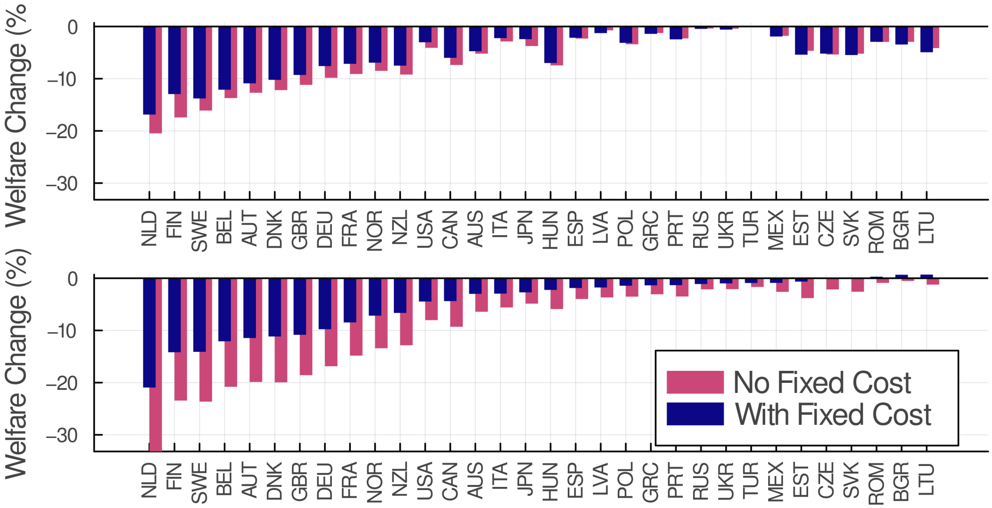
- Extensive margin: since selection cutoff falls, usually negative
- Intensive margin:
  - Trade and MP autarky shrink the profits of previously large firms engaged in these foreign activities $\rightarrow$ Relative firm sizes adjust
  - Sales-weighted in average productivity changes

MP autarky: Quantification



Left bars: results from the calibration with negative complementarities (right bars: positive complementarities)

MP autarky: Quantification



Top figure: results from calibrations with negative complementarities (bottom: positive complementarities)

To conclude

- Combinatorial discrete choice problems are common
  - Trade: multinational production, either export platforms or GVCs; firm sourcing partners; extended gravity export destinations
  - IO: input complementarity; product mix
  - (International) macro: tax avoidance and profit shifting, portfolio choice
  - Spatial economics: transport networks; real estate choices
  - ...
- We develop a new approach to CDCPs
  - With negative or positive complementarities
  - Policy function solution for aggregation
- Julia package: `CDCP.jl`

Part III

Appendix

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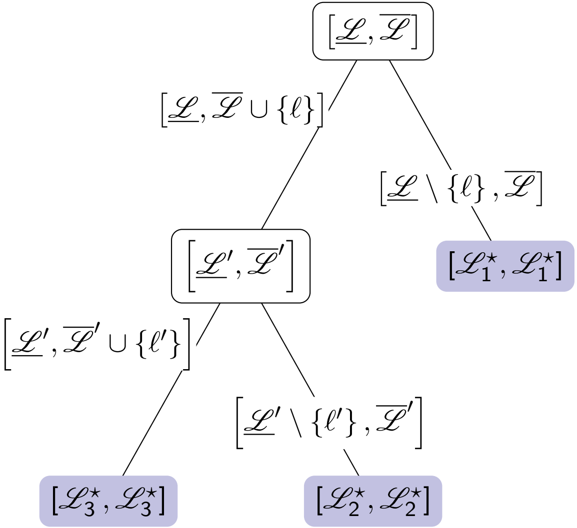
Branching tree

- convergence when bounding pair coincides on each branch
- branching collects all fixed points of Φ

invariant to the items selected and order

back



Cost minimization for each destination n

- the unit cost of producing an input υ at location $\ell \in \mathscr{L}$ then delivering it to market n is

$$\gamma_{i\ell} \frac{w_\ell}{z \varphi_\ell(\upsilon)} \tau_{\ell n}$$

$\gamma_{i\ell}$	arms-length iceberg cost of MP
w_ℓ	labor cost in production location
z	firm productivity
$\varphi_\ell(\upsilon)$	location-input shifter
$\tau_{\ell n}$	iceberg cost of trade

- tractable export platforms: for each destination n and input υ, the firm chooses the least-cost production location

Cost minimization for each destination n

- negative complementarity: marginal cost declines in $\mathscr{L}$, but decreasingly so (cannibalization)
  Antràs, Fort, and Tintelnot 2017; Tintelnot 2017

$$c_{in}(\mathscr{L};z) = \left[\int_{\varphi} \left(\min_{\ell \in \mathscr{L}} \gamma_{i\ell} \frac{w_\ell}{z\varphi_\ell} \tau_{\ell n}\right)^{1-\eta} \mathrm{d}F(\varphi;\mathscr{L})\right]^{\frac{1}{1-\eta}}$$

- closed form with Fréchet location-input draws ($\eta < \theta + 1$)
  Arkolakis, Ramondo, et al. 2018; Ramondo and Rodríguez-Clare 2013; Lind and Ramondo 2023

$$c_{in}(\mathscr{L};z) = \frac{1}{z}\Gamma \left[\sum_{\ell \in \mathscr{L}} \left(\frac{\gamma_{i\ell} w_\ell \tau_{\ell n}}{T_\ell}\right)^{-\theta}\right]^{-\frac{1}{\theta}}$$

back

General demand function

- sufficient condition for supermodularity

$$\underbrace{\varepsilon_D}_{\text{demand elasticity}} \times \underbrace{\frac{\mathrm{d}\ln p}{\mathrm{d}\ln c}}_{\text{passthrough}} \geq \underbrace{\theta}_{\text{cannibalization}} + 1$$

  - compares (positive) demand-side complementarity with (negative) supply-side complementarity
  - sufficient condition for submodularity: flip the sign
- flexible framework for discrete decisions and complementarities

back

Policy function: Japan

- with negative complementarities

```
[-Inf, 0.65]    +String[]
[0.65, 3.14]    +["JPN"]
[3.14, 3.302]   +["ROM"]
[3.302, 3.351]  +["ITA"]
[3.351, 3.403]  +["GBR"]  -["ITA", "ROM"]
[3.403, 3.574]  +["ITA"]
[3.574, 3.631]  +["ROM"]
...
```

- with positive complementarities

```
[-Inf, 0.666]   +String[]
[0.666, 4.253]  +["JPN"]
[4.253, 4.354]  +["DEU", "GBR", "FRA", "ITA", "POL", "ROM"]
...
```

back

Aggregate conditions

- Free entry. Require f_i^e to draw productivity

$$w_i f_i^e = \frac{1}{\sigma}\sum_n X_n \int \left(\frac{zP_n}{\mu}\right)^{\sigma-1} \Theta_{in}\left(\mathscr{L}_i^\star(z)\right)^{\frac{\sigma-1}{\theta}} \mathrm{d}G_i(z) - \int \sum_{\ell\in\mathscr{L}_i^\star(z)} w_\ell f_{i\ell} \mathrm{d}G_i(z)$$

- Price index. Aggregates over all firm origins i

$$P_n^{1-\sigma} = \sum_i M_i \int \left(\frac{z}{\mu}\right)^{\sigma-1} \Theta_{in}\left(\mathscr{L}_i^\star(z)\right)^{\frac{\sigma-1}{\theta}} \mathrm{d}G_i(z)$$

Aggregate conditions

- Labor market clearing. Inelastically supplied H_ℓ units

$$
\begin{aligned}
w_\ell H_\ell = \frac{\sigma-1}{\sigma} \sum_{i,n} X_n M_i \int & \frac{\mathbf{1}^\star_{i\ell}(z)\left(w_\ell \gamma_{i\ell}\tau_{\ell n}/T_\ell\right)^{-\theta}}{\sum_{k\in\mathscr{L}_i^\star(z)}\left(w_k\gamma_{ik}\tau_{kn}/T_k\right)^{-\theta}} \\
& \times \left(\frac{zP_n}{\mu}\right)^{\sigma-1} \Theta_{in}\left(\mathscr{L}_i^\star(z)\right)^{\frac{\sigma-1}{\theta}} \mathrm{d}G_i(z) \\
& + \sum_i M_i \int \mathbf{1}^\star_{i\ell}(z) w_\ell f_{i\ell} \mathrm{d}G_i(z) + M_\ell w_\ell f^e_\ell
\end{aligned}
$$

- Balance of payments.

$$X_n = w_n H_n$$

back

Quantification

- Parameterization.
  - $g_i(\cdot) \sim$ Pareto with shape ζ and minimum $\underline{z}_i$
  - bilateral trade, MP, and fixed costs with gravity variables
- Calibration strategy.

Parameter	Target
σ	set to 4
	Arkolakis, Ramondo, et al. 2018; Head and Mayer 2019
ζ	firm sales tail
	Arkolakis 2010
$T_\ell, \underline{z}_i$	GDP, total foreign MP outgoing
f_i, f_i^e	enterprise survival rate, count
$\tau_{\ell n}, \gamma_{i\ell}, \nu_{i\ell}$	trade, MP, and affiliate flow
	details

back

Bilateral costs

- Parameterization. Gravity variables $v \in \{\log \text{dist}, \text{COL}, \text{BOR}, \text{COM}\}$
  Conte, Cotterlaz, Mayer, et al. 2023

$$\log \tau_{\ell n} = \sum_v \kappa_\tau^v v_{\ell n} + \mathbf{1}[\ell \neq n]\overline{\tau}_n + \log(1 + t_{\ell n})$$

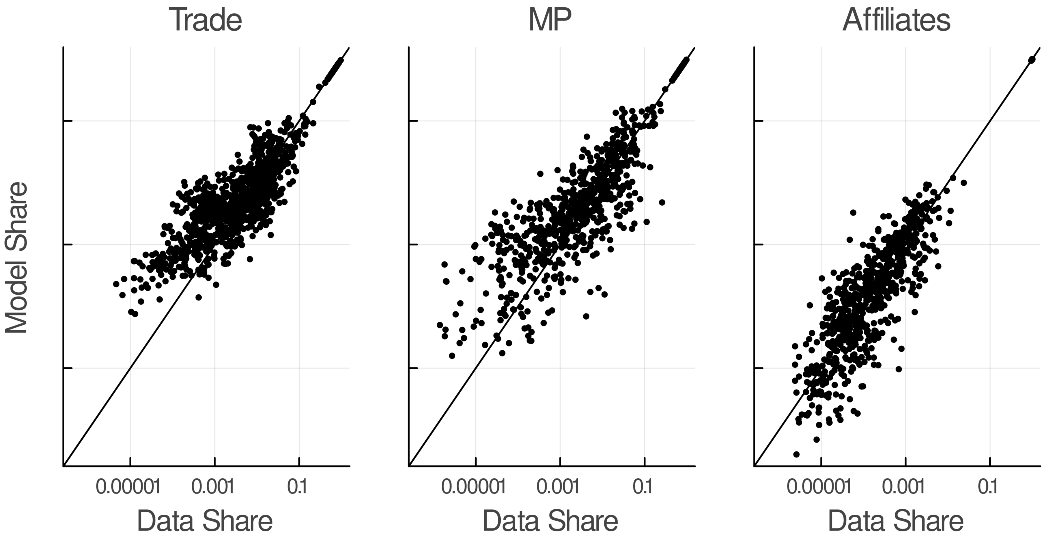
$$\log \gamma_{i\ell} = \sum_v \kappa_\gamma^v v_{i\ell} + \mathbf{1}[i \neq \ell]\overline{\gamma}_n$$

$$\log f_{i\ell} = \sum_v \kappa_f^v v_{i\ell} + \mathbf{1}[i \neq \ell]\overline{f}_n$$

- Match aggregate flows.
  - $\kappa_\tau^v, \kappa_\gamma^v, \kappa_f^v$ corresponding coefficient on gravity variables in trade, MP, and affiliate regressions
  - $\overline{\tau}_n, \overline{\gamma}_\ell, \overline{f}_\ell$ own shares of "trade" (absorption), "MP" (domestic production), and affiliates (domestic production locations)

back

Micro-data not required



Calibration matches aggregates shares (with positive complementarities)